1) M.T. Nair.

→ Euclidean norm

→ Cauchy Schwartz inequality

$$\sum_{i=1}^{n} |\alpha_i \beta_i| \le \left(\sum_{i=1}^{n} |\alpha_i|^2\right)^{1/2} \left(\sum_{i=1}^{n} |\beta_i|^2\right)^{1/2}$$

$$ab \le \frac{a^2+b^2}{2}.$$

→ p- norm

→ Holder's inequality

$$\sum |\alpha_i \beta_i| \le \left(\sum |\alpha_i|^p\right)^{1/p} \left(\sum |\beta_i|^q\right)^{1/q} \qquad \frac{1}{p}+\frac{1}{q}=1.$$

→ sequence $x_n : \mathbb{N} \longrightarrow \mathbb{K}$ field.

→ A^B : set of all fns from B to A.

$\mathbb{K}^{\mathbb{N}}$: set of all sequences with entries from $\mathbb{K}$.

→ linear space.

$x = (\alpha_1, \alpha_2, \cdots)$

$$\|x\|_\infty = \sup\{|\alpha_j| : j = 1,2,\cdots\}$$

$$\|x\|_1 = \sum_{j=1}^{\infty} |\alpha_j|. \qquad \|x\|_2 = \left(\sum_{j=1}^{\infty} |\alpha_j|^2\right)^{1/2}$$

$$\sum_{i=1}^{n} |xy| \le \|x\|_2 \|y\|_2$$

$n \to \infty$ $$\sum_{i=1}^{\infty} |xy| \le \|x\|_2 \|y\|_2. \quad \text{proof.}$$

→ $\|x+y\|_p \le \|x\|_p + \|y\|_p$ Minkowski

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$\|\alpha x\|_2 = |\alpha| \, \|x\|_2$

$x \mapsto \|x\|_2$ is a norm on $\ell^2 = \{ x \in \mathbb{K}^{\mathbb{N}} : \sum_{i=1}^{\infty} |x(i)|^2 < \infty \}$

$\ell^1 = \{ x : \sum_{i=1}^{\infty} |x(i)| < \infty \}$

$\ell^\infty = \{ x : \sup_{i \in \mathbb{N}} |x(i)| < \infty \}$

→ $|xy| \leq \|x\|_p \|y\|_q$. $\frac{1}{p} + \frac{1}{q} = 1$.

$\begin{cases} \|x+y\|_p \leq \|x\|_p + \|y\|_q \\ \|\alpha x\|_p = |\alpha| \, \|x\|_p \end{cases}$ $\Rightarrow$ $\ell^p = \{ x \in \mathbb{K}^{\mathbb{N}} : \sum_{i=1}^{\infty} |x_i|^p < \infty \}$

is a linear space

→ ℓ^∞

→ $X = C[a,b]$

$$\|x\|_1 = \int_a^b |x(t)| \, dt.$$

$$\|x\|_\infty = \sup_{a \leq t \leq b} |x(t)| = \max_{a \leq t \leq b} |x(t)|.$$

→ Holder's ineq.

$$\int_a^b |x(t) y(t)| \, dt \leq \left(\int_a^b |x(t)|^p dt \right)^{1/p} \left(\int_a^b |y(t)|^q dt \right)^{1/q}.$$

True for $x, y \in \mathcal{R}[a,b]$ Riemann integrable fn's.

$$f_n = \begin{cases} 1, & x \in \{r_1, r_2, \dots r_n\} \\ -1, & x \in [a,b] \setminus \{r_1, r_2, \dots r_n\} \end{cases} \longrightarrow \begin{cases} 1, & x \in \mathbb{Q} \\ -1, & x \in [a,b] \setminus \mathbb{Q} \end{cases}$$

Pointwise approximation.

→ Holder's & Minkowski are true for all Lebesgue measurable f^{n}s $\int |f|^p \, d\mu < \infty$.

$\mathcal{L}^p[a,b] = \{ f : f$ is Leb. measurable $\ni \int_a^b |f|^p \, d\mu < \infty \}$

$\|x\|_p = 0 \Leftrightarrow x = 0$ a.e.

$Z_p = \{ f \in \mathcal{L}^p[a,b] , f = 0 \text{ a.e} \}$

On $\mathcal{L}^p[a,b]$ $\quad f \sim g \Leftrightarrow f - g \in Z_p$.

$L^p[a,b] = \mathcal{L}^p[a,b] / Z_p \qquad \sim$ is an equivalence relation

$[f]$ $\quad \|f\|_p = 0 \Rightarrow f = 0$ a.e $\Leftrightarrow [f] = [0]$.

* Norm on a linear space induces a metric
$d(x,y) = \|x - y\|$.

* A metric which is bounded is never induced by a norm.

* Note that if X is a non-zero nls with norm $\|\cdot\|$ the norm $x \longmapsto \|x\|$ is not a bdd f^n.

$x_0 \neq 0 \in X \Rightarrow \|n x_0\| = n\|x_0\|, n$ is un bdd sequence.

for bdd metric $\exists M$, $d(x,y) \leq M$. $\forall x$

→ Banach space.

→ Every finite dim. nls. is a Banach space.

$\text{span}\{u_1, u_2, \ldots, u_K\} \xrightarrow{T} \mathbb{K}^K$.

$x = \alpha_1 u_1 + \cdots + \alpha_n u_K \longrightarrow (\alpha_1, \ldots, \alpha_K)$

→ $\mathcal{P}_K$: {poly. with degree $\leq K$}

$\{1, t, t^2, \ldots, t^K\}$

$\rightarrow$ $(\Omega, d) \xrightarrow{T} (\tilde{\Omega}, \tilde{d})$

Metric space — Complete metric space.

- $\tilde{d}(T(x), T(y)) = d(x, y)$.
- $R(T)$ dense in $\tilde{\Omega}$

then $\tilde{\Omega}$ is a completion of Ω.

The completion is unique upto isometry.

$\rightarrow$ Given any nls $(X, \|\cdot\|_X)$ $\exists$ a Banach space $(Y, \|\cdot\|_Y)$ and a linear isometry $T: X \longrightarrow Y$ $\ni$ $R(T)$ is dense in Y. ~~Further if~~ $\cancel{(Z, \|\cdot\|_Z)}$

Such a Banach space Y is called a completion of $(X, \|\cdot\|_X)$.

If $(Z, \|\cdot\|_Z)$ is also a completion of $(X, \|\cdot\|_X)$ then Y & Z are linearly isometric.

R.R.

$T: X \longrightarrow Y$.

$\{v_1, v_2, \dots, v_n\} \longrightarrow \{w_1, \dots, w_m\}$

Matrix representation. $[a_{ij}]$

$$[a_{ij}]_{m\times n} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix} \Rightarrow T(x) = \sum_{i=1}^{m} y_i w_i .$$

When we say a metric is induced by a norm?

1) Scalar multiplication should be continuous.

$T: \underset{n}{X} \longrightarrow \underset{m}{Y}$

T is 1-1 $\Leftrightarrow$ $N(T) = \{0\}$.

$\Leftrightarrow$ T is onto

$\Leftrightarrow$ T is invertible.

T is singular $\Rightarrow \exists\, x \neq 0 \ni T(x) = 0$.

$\det(T_B) = 0$.

→ consider $\begin{bmatrix} a_{ij} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

→ $f: I \longrightarrow \mathbb{R}$, $x_0 \in \operatorname{Int}(I)$ $\quad f'(x_0) = \lim_{x \to x_0} \dfrac{f(x) - f(x_0)}{x - x_0}$.

$f: I \longrightarrow \mathbb{R}^n$.

$$f'(x_0) = \lim_{x \to x_0} \frac{\vec{f}(x) - \vec{f}(x_0)}{x - x_0}.$$

$f'(x_0)$ exists iff $f_i'(x_0)$ exists $\forall i$

let $R(x, x_0) = \dfrac{f(x) - f(x_0)}{x - x_0}$

If $R(x, x_0) - f'(x_0) \longrightarrow 0$ as $x \to x_0$ then f is diff.

$\Rightarrow$ $f(x) - f(x_0) = \left[R(x, x_0) + f'(x_0)\right](x - x_0)$.

$T: \mathbb{R}^n \longrightarrow \mathbb{R}$

$f: S \longrightarrow \mathbb{R}$. , $S \subset \mathbb{R}^n$

$f(\vec{x}_0 + \vec{x}) - f(\vec{x}_0) = T(\vec{x}) + \|x - x_0\|\, R(x, x_0)$ → how ?

$\forall x \in B_r(x_0)$ where $R(x, x_0) \longrightarrow 0$ as $x \to x_0$.

$$T = \begin{bmatrix} D_1 f \\ D_2 f \\ \vdots \\ D_n f \end{bmatrix} \qquad D_j f = \frac{\partial f}{\partial x_j} = \lim_{t \to 0} \frac{f(\vec{x_0} + t\vec{e_j}) - f(x_0)}{t}$$

→ Chain rule

$\vec{r} : I \to \mathbb{R}^n \qquad \phi : \mathbb{R}^n \to \mathbb{R} \qquad \underbrace{\phi \circ \vec{\gamma}}_{=f} : I \to \mathbb{R}$

$$f'(t) = \nabla \phi(\vec{r}(t)) \cdot \vec{r}'(t).$$

→ Mean value theorem

$f : [a,b] \to \mathbb{R}^n$

$$\| f(b) - f(a) \| \leq \| f'(t) \| (b-a).$$

$\phi : S \subset \mathbb{R}^n \to \mathbb{R}. \qquad \vec{a} \in S$

$$\phi(\vec{a} + \vec{y}) - \phi(\vec{a}) = \nabla \phi(\bar{a} + t\bar{y}) \cdot \bar{y} \text{ for some } t \in (0,1).$$

→ Inverse function theorem.

→ let $f : S \subset \mathbb{R}^n \to \mathbb{R}^n \qquad f \in C^1(S).$

$\exists \; J_f(a) \neq 0$

$x = (x_1, x_2, \ldots, x_n)$

$f = (f_1, \ldots, f_n)$

$$J_f = \begin{pmatrix} & & D_i f_1 & & \\ & & D_i f_2 & & \\ & & \vdots & & \\ D_1 f_j & D_2 f_j & D_j f_j & \cdots & D_n f_j \\ & & \vdots & & \\ & & D_i f_n & & \end{pmatrix}$$

Define $F : S^n \to \mathbb{R}^{F} \qquad S \subset \mathbb{R}^n.$

meaning?

$$F(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n) = \det [(D_j f_i)(\bar{x_j})]_{n \times n}.$$

F is continuous. $\qquad f_i : S \to \mathbb{R}.$

$P_j : S^n \to S$ by

$$P_j(\bar{x}_1, \ldots \bar{x}_n) = \bar{x}_j$$

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$\left[\left(D_j f_i\right) \circ P_j\right] : S^n \longrightarrow \mathbb{R}.$

$F(\bar{a}, \bar{a}, \cdots \bar{a}) \neq 0 \Rightarrow \exists$ a nbhd N of $(a, a, \cdot\cdot \bar{a}) \ni$

$F \neq 0$ on N.

Using MVT.

pr $f : B_\varepsilon(\bar{a}) \longrightarrow f(B_\varepsilon(a))$ is 1—1.

$$0 = f_i(\bar{x}) - f_i(\bar{y})$$

$$= \nabla f_i(\xi_i)(\bar{x} - \bar{y}) \quad \forall i = 1, 2, \cdots n.$$

$$= \sum_{j=1}^{n} (D_j f_i)(\xi_i)(x_j - y_j).$$

contradict^n ?

Same as $\left[\, D_j f_i \,\right] \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix} \begin{bmatrix} x_1 - y_1 \\ \vdots \\ x_n - y_n \end{bmatrix} = 0.$

<u>claim</u> For a given $\vec{v} \in B_\varepsilon(\bar{a})$ and $\eta > 0$ is $\ni$

$\overline{B_\eta(\vec{v})} \subseteq B_\varepsilon(\bar{a})$ then $\exists \; \delta > 0 \ni$

$B_\delta(\vec{f}(\vec{v})) \subseteq f(B_\eta(\vec{v}))$

→ stationary pt
→ pt of inflection

→ boundary

Conditions for extrema @ bdry.

$\phi : \overline{B_\eta(v)} \longrightarrow \mathbb{R}.$

$\phi(u) = \| f(\bar{u}) - \bar{y} \|^2 \qquad \bar{y} \in B_\delta(f(\bar{v}))$

$\overline{B_\eta(v)}$ is compact.

$\therefore \phi$ min attains @ w. We prove w is interior pt.

ie, $w \notin \overline{B_\eta(v)} \setminus \partial B_\eta(\bar{v})$

ϕ has minimum at $\overline{w}$.

$(\nabla \phi)(\overline{w}) = 0 \Rightarrow (D_j \phi)(\overline{w}) = 0 \quad \forall j = 1, \ldots, m.$

$D_j \phi(\overline{w}) = \frac{\partial}{\partial x_j} \left(\sum_{i=1}^{n} f_i(\overline{x}) - \overline{y} \right)^2 (\overline{w}) = 0.$

let $g : f(B_\varepsilon(\overline{v})) \longrightarrow B_\varepsilon(\overline{v})$ be the inverse of f.

let $\overline{b} \in f(B_\varepsilon(\overline{v}))$

$\Rightarrow b = f(x)$ for some $x \in B_\varepsilon(\overline{v})$

$$\frac{g(b + te_k) - g(b)}{t}.$$

→ Implicit function theorem.

$f : U \subset \mathbb{R}^{n+m} \longrightarrow \mathbb{R}^m, \quad f \in C^1$

$\overline{z} = (\overline{x}, \overline{y}), \quad \overline{x} \in \mathbb{R}^n, \ \overline{y} \in \mathbb{R}^m \qquad f(x_1, x_2, \ldots, x_n, y_1, \ldots, y_m)$

$\frac{\partial f}{\partial \overline{y}}(\overline{a}, \overline{b}) : \mathbb{R}^m \longrightarrow \mathbb{R}^m$ derivative of f at $(\overline{a}, \overline{b})$

$a \in \mathbb{R}^n, \ b \in \mathbb{R}^m.$

then $\exists$ open subset A of $\mathbb{R}^n$, W of $\mathbb{R}^m$ and

$g : A \to \mathbb{R}^m \ni$

$V \subseteq U.$

1) g is a C^1-function.

2) $\{(\overline{x}, \overline{y}) \in V : f(\overline{x}, \overline{y}) = 0\} = \{(\overline{x}, g(\overline{x})) : \overline{x} \in A\}$

3) $g'(x) = -\left[\frac{\partial f}{\partial \overline{y}} (\overline{x}, g(\overline{x})) \right]^{-1} \left[\frac{\partial \overline{f}}{\partial \overline{x}} (\overline{x}, g(\overline{x})) \right]$

Just like,

$f(x,y) = 0$.

$$\left[\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = 0 . \quad \frac{dy}{dx} = \frac{-\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} \right]$$

Introduce $\bar{F}(x,y) = (\bar{x}, f(\bar{x},\bar{y}))$ $\quad F(a,b) = (a,0) \in W$.

$\bar{F}$ is c^1, $\because$ $\bar{x}$ & f are c^1.

$F : U \subset \mathbb{R}^{n+m} \longrightarrow \mathbb{R}^{n+m}$ $\quad \bar{F} = (x_1, x_2, \dots, x_n, f_1, \dots, f_m)$

Diff w.r.t x for n coordinates and y for rest.

$$[F'(\bar{a},\bar{b})] = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\ D_1 f_1 & \cdots & & D_n f_1 & & & \\ D_1 f_2 & & & \vdots & & \frac{\partial f}{\partial \bar{y}}(\bar{a},\bar{b}) & \\ D_1 f_m & & & D_n f_m & & & \end{bmatrix}$$

$$|[F'(a,b)]| = \left| \left[\frac{\partial f}{\partial \bar{y}} (\bar{a}, \bar{b}) \right] \right| \neq 0 .$$

$\Rightarrow$ We can apply inverse f^n then,

$\Rightarrow \exists$ open $V \subset U$ and open $W \subset \mathbb{R}^{n+m}$ $\ni$

$\ni$ $F : V \longrightarrow W$ is a bijection &

$F^{-1} = G : W \longrightarrow V$ is a c^1 map.

Let $A = \{ x \in \mathbb{R}^n \,/\, (\bar{x}, \bar{0}) \in W \}$

A is open $\left(A = h^{-1}(W), \ h(\bar{x}) = (\bar{x}, 0), \ \forall x \in \mathbb{R}^n \right)$

Define $g : A \longrightarrow \mathbb{R}^m$ by

$g(x) = P_2 (G(\bar{x}, 0))$ $\quad \forall \bar{x} \in A$.

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$(\bar{x},\bar{y}) \in V$ ~~$\ni$~~ $f(x,y)=0 \iff (x,y) \in V \iff (\bar{x},y) \in V,\ F(x,y)=(x,0)$

$\iff (x,0) \in W$ and $G(x,0)=(x,y)$

$\iff x \in A$ and $P_2(G(x,0))=\bar{y}$. ie, $g(\bar{x})=\bar{y}$.

② condition is proved.

Define $\phi : A \longrightarrow \mathbb{R}^{n+m}$ by $\phi(x)=(x,g(x))\ \forall x \in A$

observe that

$f \circ \phi = 0$ on A.

ie; $f(x,g(x)) = 0$ on A.

$[f'(\phi(x))]^{*} \circ \phi'(\bar{x}) = 0.$

$$\left[\frac{\partial f}{\partial x}(\bar{x},g(\bar{x})) \;\middle|\; \frac{\partial f}{\partial \bar{y}}(\bar{x},g(\bar{x})) \right] \begin{bmatrix} I_{n\times n} \\ [g'(x)]_{m\times n} \end{bmatrix}_{(m+n)\times n} = 0.$$

$$\left[\frac{\partial f}{\partial x}(x,g(x))\right] + \frac{\partial f}{\partial y}(x,g(x))[g'(x)] = 0$$

$\Rightarrow$ ~~$g'(x) = \left[\frac{\partial f}{\partial x}(x,g(x))\right]\left[\frac{\partial f}{\partial y}(x,0\right.$~~

$$\Rightarrow g'(x) = -\left[\frac{\partial f}{\partial y}(x,g(x))\right]^{-1}\left[\frac{\partial f}{\partial x}(x,g(x))\right]$$

$\longrightarrow$ Hessian

$f(x_1,\dots,x_n) \in \mathbb{R}$.

$$H_f(a) = \left[(D_{ij} f)(a) \right]_{n\times n}$$

$$f(\bar{a}+\bar{y}) - f(\bar{a}) = \underset{(=0)}{(\nabla f)(a)(\bar{y})} + (H_f)(a)\|y\|^2 + (\|y\|^3)(\quad)\dots$$

(If a is a stationary pt)

Lagrange's Multiplier Method.

let $m<n$ & rank $\left[\frac{\partial\psi}{\partial x}(\bar{a})\right]$ is m

Optimize $\phi(x_1, \cdots, x_n)$

subject to $\psi_1(x_1 \ldots, x_n) = 0$

$\vdots$

$\psi_m(x_1, \ldots, x_n) = 0$

ϕ, ψ_i are $C'(S)$ fn's

S is open

If $w = (w_1, \cdots, w_n)$ is a stationary pt. of ϕ satisfying $\psi_i(\bar{w}) = 0$ then $\bar{w}$ is a stationary point for $F = \phi - \sum_{i=1}^{m} \lambda_i \psi_i$ for some scalars $\lambda_1, \cdots \lambda_m$

or w is a soln for $D_j F = 0$ $i = 1, 2, \cdots, n$; where

$\psi = (\psi_1, \ldots \psi_m)$

Rank $[D_j \psi_i]_{m\times n} = m$.

let $[D_{j_1}\psi_i]_{m\times 1}, [D_{j_2}\psi_i]_{m\times 1} \cdots [D_{j_m}\psi_i]_{m\times 1}$ be independent.

$$\det\left[\frac{\partial\psi}{\partial\bar{y}}(\bar{w})\right]_{m\times m} \neq 0.$$

$w = (\bar{u}, \bar{v})$ $\bar{\psi}_i(W) = 0 \Rightarrow \bar{\psi}(\bar{u}, \bar{v}) = 0$

$\bar{u} \to \mathbb{R}^{n-m}$, $\bar{v} \to \mathbb{R}^m$

$\therefore$ By Implicit fn thm,

$\exists$ an open subset $T \subseteq S$ and an open subset $E \subseteq \mathbb{R}^{n-m}$

$\left(f : U \subseteq \mathbb{R}^{n+m} \to \mathbb{R}^m, \ \psi : S \to \mathbb{R}^m\right)$ and a C' fn

$\sigma : E \to \mathbb{R}^m \ni$

1) $\{(x,y) \in T : \psi(x,y) = 0\} = \{(x, \sigma(x)) : \bar{x} \in E\}$

$\bar{\psi}(x, \sigma(x)) = 0 \quad \forall x \in E.$

$D_j \left[\bar{\Psi}_i \cdot (x, \sigma(x))\right] = 0 \quad \forall \bar{x} \in E.$

$$= \nabla \bar{\Psi}_i (x, \sigma(x)) \Big[\underbrace{0\,0 \cdots 1\,0 \cdots 0}_{n-m} \Big| \underbrace{D_j\sigma_1 \;\; D_j\sigma_2 \cdots D_j\sigma_m}_{m}\Big]$$

$$\left((D_1\Psi_i)(x,\sigma(x)), (D_2\Psi_i)(x,\sigma(x)) \ldots, D_n(\Psi_i)(x,\sigma(x))\right)\Big[0\,0 \cdots 1\,0 \cdots 0 \Big| D_j\sigma_1 \cdots D_j\sigma_m\Big]$$

$$0 = (D_j\Psi_i)(x,\sigma(x)) + \sum_{k=1}^{m} \left(D_{(n-m)+k}\Psi_i\right)(x,\sigma(x))\,(D_j\sigma_k)(\bar{x})$$

$\bar{x} = (\bar{u},\bar{v})$

$$0 = D_j\Psi_i(w) + \sum_{k=1}^{n} \left(D_{n-m+k}\Psi_i\right)(w)\,(D_j\sigma_k)(\bar{u}) \qquad w = (u,v)$$

$\because$ $\bar{w}$ is a stationary point for ϕ satisfying $\bar{\Psi}(\bar{x}) = 0$

$\Rightarrow$ $\bar{w}$ is a stationary point for $\phi(\bar{x}, \sigma(x))$

$m \times (m+1) \Big[$

$(m+1)\times 1$

$m \times 1$.

MTN.

* Every f.d. nls is complete.

In particular, every f.d. subspace of a nls is complete and hence closed.

Theorem Let X be a nls, Y be a complete subspace and Z be a f.d. subspace of X.

Then $Y + Z$ is complete.

$$Y + Z = \{y + z : y \in Y,\, z \in Z\}.$$

Proof Suppose Z is 1D. i.e, $Z = [\{x_0\}]$ $\quad x_0 \neq 0$.

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$Y+Z = \{ y + \alpha x_0 : y \in Y, \alpha \in K \}$

$x_n \in Y+Z \iff x_n = y_n + \alpha_n x_0.$

$$\|x_n - x_m\| = \|(y_n - y_m) + (\alpha_n - \alpha_m)x_0\|$$
$$\geq \text{dist}\left((\alpha_n - \alpha_m)x_0, Y \right)$$
$$= |\alpha_n - \alpha_m| \, \text{dist}(x_0, Y)$$

$\text{dist}(x_0,Y) > 0$ $\because Y$ is complete $\Rightarrow Y$ is closed.

$\Rightarrow \alpha_n$ is a Cauchy sequence in $\mathbb{K}$.

$\Rightarrow \exists \, \alpha \in \mathbb{K} \ni \alpha_n \longrightarrow \alpha$

$y_n = x_n - \alpha_n x_0$ is Cauchy. $\Rightarrow y_n \longrightarrow y$

$\Rightarrow x_n \longrightarrow y + \alpha x_0 . \in Y+Z.$

* Heine - Borel Property.

every closed & bdd subset of $\mathbb{K}^n$ is compact wrt $\|\cdot\|_\infty$.

$\because$ $C_p' \|x\|_p \leq \|x\|_\infty \leq C_p \|x\|_p$

* let X be n dim nls; let $E = \{u_1, \cdots, u_n\}$ be a basis of X,

$x \in X \Rightarrow (\alpha_1, \cdots, \alpha_n) \in \mathbb{K}^n \ni$

$$x = \sum_{j=1}^{n} \alpha_j u_j$$

$$\|x\|_E = \max_{1 \leq j \leq n} |\alpha_j|$$

$$X \xrightarrow{T} \mathbb{K}^n$$
$$\sum_{j=1}^{n} \alpha_j u_j \longmapsto (\alpha_1, \cdots, \alpha_n)$$

T is 1-1, onto isometry.

$\|T(x)\|_\infty = \|x\|_E$.

$\Rightarrow$ Using Heine-Borel thm on $\mathbb{K}^n$, we have Heine Borel thm on $(X, \|\cdot\|_E)$.

* Let X be a f.d. nls with $\|\cdot\|$. If $\|\cdot\|$ is equivalent to $\|\cdot\|_E$ for some basis E, then Heine-Borel thm is true on $(X, \|\cdot\|)$.

Let X be a f.d. nls and by $E = \{u_1, u_2 \cdots u_n\}$ be a basis of X. Then any $x \in X$ can be written uniquely as $x = \sum_{j=1}^{n} \alpha_j u_j$

$$\|x\| \leq \sum_{j=1}^{n} |\alpha_j| \, \|u_j\| \leq \|x\|_E \sum_{j=1}^{n} \|u_j\|.$$

$$\Rightarrow \quad \|x\| \leq \sum_{j=1}^{n} \|u_j\| \, \|x\|_E .$$

$$\|x\| = \|\alpha_1 u_1 + \cdots + \alpha_n u_n\|$$

$$= \| \alpha_j u_j + \underbrace{\sum_{i \neq j} \alpha_i u_i}_{\in \text{Span}\{u_i, i \neq j\} = X_j} \|$$

$$\geq \text{dist}(\alpha_j u_j, X_j)$$

$$= |\alpha_j| \, \text{dist}(u_j, X_j) \qquad \because u_j \notin X_j$$

$$\geq |\alpha_j| \min_{1 \leq k \leq n} (u_k, X_k)$$

$$= |\alpha_j| \, \beta \quad \forall j$$

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$\Rightarrow \|x\| \geq \max_j |\alpha_j| \beta = \beta \|x\|_E$.

$\|x\| \geq \beta \|x\|_E$

* Let X be a ~~f.d.~~ nls.

1) X is f.d.

2) Every closed and bdd set in X is compact

$\Rightarrow \{x \in X : \|x\| \leq 1\}$ is compact.

3) $\{x \in X : \|x\| = 1\}$ is compact. ($\because$ closed subset of unit ball

Theorem (Best approximation thm)

Let X be a nls and X_0 be a f.d. subspace of X. Then for every $x \in X$, $\exists\, x_0 \in X_0 \ni$

$$\text{dist}(x, X_0) = \|x - x_0\|$$

Pf

If $x \in X_0$, take $x_0 = x$. Suppose $x \notin X_0 \Rightarrow d := \text{dist}(x, X_0) > 0$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| \quad \inf_{u \in X_0} \|x - u\|$

$\Rightarrow \exists\, (u_n) \in X_0 \ni \|x - u_n\| \longrightarrow d$.

$\Rightarrow u_n$ is bdd. ($\because \|u_n\| \leq \|u_n - x\| + \|x\|$)

$\Rightarrow \exists\, (u_{k_n}) \longrightarrow u$ in X_0.

$\Rightarrow \|x - u_{k_n}\| \longrightarrow \|x - u\|$.

$\quad\downarrow d$

$\Rightarrow d = \|x - u\|$.

Corollary Let X be a nls; X_0 be a f.d. proper subspace. Then $\exists\, \tilde{x} \in X \ni \|\tilde{x}\| = 1$ & $\text{dist}(\tilde{x}, X_0) = 1$.

let $x \in X \setminus X_0$.

$\Rightarrow \exists \; x_0 \in X_0 \ni \text{dist}(x, X_0) = \|x - x_0\|$

$$\tilde{x} = \frac{x - x_0}{\|x - x_0\|} \Rightarrow \|\tilde{x}\| = 1$$

$$\text{dist}(\tilde{x}, X_0) = \text{dist}\left(\frac{x - x_0}{\|x - x_0\|}, X_0\right)$$

$$= \frac{1}{\|x - x_0\|} \text{dist}(x - x_0, X_0)$$

*. Suppose $S := \{x : \|x\| = 1\}$ is compact.

(3) $\Rightarrow$ (1)

Suppose X is not f.d.

let $u_1 \in X$. let $X_1 := \text{Span}\{u_1\}$.

$\because$ X_1 is a f.d. proper subspace of X

by corollary, $\exists \; u_2 \in X \ni \|u_2\| = 1$ & $\text{dist}(u_2, X_1) = 1$

let $X_2 = \text{sp}\{u_1, u_2\}$

$\exists \; u_3 \in X \ni \|u_3\| = 1$ & $\text{dist}(u_3, X_2) = 1$

Ily

$\exists \; (u_n) \in X \ni \|u_n\| = 1 \;\; \forall n$ & $\text{dist}(u_n, X_{n-1}) = 1 \;\; \forall n = 2, 3.$

$\Rightarrow$ for $n > m$

$$\|u_n - \underbrace{u_m}_{\in X_{n-1}}\| \geq \text{dist}(u_n, X_{n-1}) = 1.$$

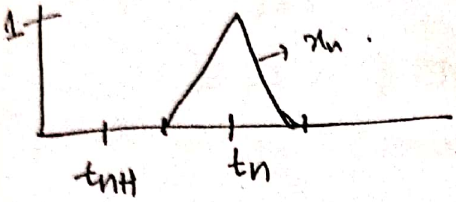
$\Rightarrow (u_n)$ doesn't have a Cauchy subsequence.

$\Rightarrow u_n \in S$ doesn't have a convergent s.

$\Rightarrow$ S is not compact.

* Thm X is a nls. X is complete $\Leftrightarrow$ $\{x : \|x\| \le 1\}$ complete.

consider $C[a,b], \|\cdot\|_\infty$. $t_{n+1} = a + \frac{b-a}{n}$



$x_n \to$ Hat f's

$\|x_n - x_m\|_\infty = 1 \ \forall n \neq m$ $\qquad \|x_n\|_\infty = 1$.

* Innerproduct spaces.

$\|x\| := \langle x, x\rangle^{1/2}$.

$\rightarrow$ X is an ips.

$x \perp y \Rightarrow \|x \pm y\|^2 = \|x\|^2 + \|y\|^2$.

$\rightarrow$ Schwarz inequality $|\langle x, y\rangle| \le \|x\| \|y\|$. Norms not from ip

Pf let $x \in X$,

$\rightarrow$ Orthogonal set $\langle x, y\rangle = 0 \ \forall x, y$.

$\rightarrow$ Orthonormal set.

$\rightarrow$ orthonormal set is L.I.

$\rightarrow$ An orthonormal set is called an onb if it is a maximal orthonormal set.

$\rightarrow$ Every ips has an onb.

→ onb need not be a basis.

* Let E be an orthonormal set in an ips X.
Then E is an onb $\Leftrightarrow E^{\perp} = \{0\}$.

⇒ Suppose E is onb.

Suppose $x \in X \ni 0 \neq x \perp E$, $u = \frac{x}{\|x\|}$.

$\tilde{E} = \{u, E\}$. $\tilde{E}$ is an ons & $E \subset \tilde{E}$.

This is a contradiction.

→ Hilbert space.

ℓ^2, $(\mathbb{K}^n, \langle , \rangle)$

→ Given $\{x_1, \ldots, x_n\}$ L.I. in an ips X.

define $u_1 = x_1$ $\quad v_1 = u_1/\|u_1\|$.

$u_2 = x_2 - \langle x_2, v_1 \rangle v_1 \quad v_2 = \frac{u_2}{\|u_2\|}$.

$u_3 = x_3 - \langle x_3, v_1 \rangle v_1 - \langle x_3, v_2 \rangle v_2$.

$\text{span}\{v_1, \ldots, v_j\} = \text{span}\{u_1, \ldots, u_j\} = \text{span}\{x_1, \ldots, x_j\}$

→ Bessel's inequality

Given any ons $\{u_1, \ldots, u_n\} \subset X$.

$$\sum_{j=1}^{n} |\langle x, u_j \rangle|^2 \leq \|x\|^2.$$

~~pf~~ $$x = \left(x - \sum_{j=1}^{n} \langle x, u_j \rangle u_j\right) + \sum_{j=1}^{n} \langle x, u_j \rangle u_j$$

$\rightarrow \perp \leftarrow$

$\Rightarrow$ Pythagoras, $\|x\|^2 \geq \|\sum_{j=1}^{n} \langle x, u_j \rangle u_j\|^2 = \sum_{j=1}^{n} |\langle x, u_j\rangle|^2$.

* For ons $\{u_1, u_2, \ldots\ldots\}$

$$\sum_{j=1}^{\infty} |\langle x, u_j\rangle|^2 \leq \|x\|^2 \quad \forall x \in X.$$

* If X is a separable ips, then every ons of X is countable.

let E be an ons of X. Then $\|u-v\| = \sqrt{2}$ $\forall\, u \neq v \in E$

(Suppose E is uncountable)

$\Rightarrow$ Every dense subset of X has to be uncountable.

$\Rightarrow$ X not sep.

* Let X be an ips, E be an ons. Then for every $x \in X$, the set, $E_x := \{u \in E : \langle x, u\rangle \neq 0\}$ is countable.

Note that $E_x = \{u \in E : |\langle x, u\rangle| > 0\}$

$$= \bigcup_{n=1}^{\infty} \underbrace{\{u \in E : |\langle x, u\rangle| > \tfrac{1}{n}\}}_{E_{x,n}}$$

let $u_1, \ldots, u_k \in E_{x,n}$

$$\|x\|^2 \geq \sum_{j=1}^{k} |\langle x, u_j\rangle|^2 \geq \frac{k}{n^2}$$

$\Rightarrow k \leq n^2 \|x\|^2$ $\quad\because$ n & x are fixed

$\Rightarrow E_{x,n}$ is finite. $\quad$ k is finite.

$\Rightarrow E_x$ is countable.

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→ Theorem ~~let X be an ips and E be an ons.~~

let X be a separable Hilbert space and $E = \{u_1, u_2, \dots\}$ ons. TFAE,

1) E is an o.n.b.

2) $x = \sum_j \langle x, u_j \rangle u_j \quad \forall x \in X$ (Fourier expansion)

3) $\|x\|^2 = \sum_j |\langle x, u_j \rangle|^2 \quad \forall x \in X$ (Parseval's thm)

If E is finite, $E = \{u_1, \dots, u_n\}$ then $x_n = x$

If E is not finite,

For $n > m$, $\|x_n - x_m\|^2 = \|\sum_{j=m+1}^{n} \langle x, u_j \rangle u_j\|^2$

$= \sum_{j=m+1}^{n} |\langle x, u_j \rangle|^2 \longrightarrow 0$ as $n, m \to \infty$ by Bessel's

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* If X is f.d. then every onb is a basis.
* An onb need not be a basis.

$X = \ell^2 \qquad E = \{e_1, e_2, \dots\}$

$E^\perp \Rightarrow \langle x, e_i \rangle = 0 \ \forall i$

$\Rightarrow x(i) = 0 \ \forall i \Rightarrow x = 0$.

$E^\perp = \{0\} \quad \therefore E$ is onb.

* An onb can be a basis.

$X = C_{00}$, $E = \{e_1, e_2, \dots\}$ is an onb and also a basis.

* $\boxed{\text{In a Hilbert space } X \text{ an onb is a basis iff } X \text{ is f.d.}}$

* Let X be an infinite dim. Hilbert space and let E be an onb of X ($\Rightarrow E$ is an infinite set)

Let $\{u_1, u_2, \ldots\} \subseteq E$

basis should be countable ?

Let $\alpha_1, \alpha_2, \ldots$ Scalars $\ni \alpha_j \neq 0 \ \forall j$ & $\sum_{j=1}^{\infty} |\alpha_j|^2 < \infty$

$(\alpha_1, \alpha_2 \ldots) \in \ell^2$ $\alpha_j \neq 0 \ \forall j$

Take partial sums

Then $x_n = \sum_{j=}^{n} \alpha_j u_j$

For $n > m$ $x_n - x_m = \sum_{j=m+1}^{n} \alpha_j u_j \Rightarrow \|x_n - x_m\|^2 = \sum_{j=m+1}^{n} |\alpha_j|^2 < \infty$

$\Rightarrow$ (x_n) is a cauchy sequence.

$\Rightarrow$ $\exists x \in X \ni \|x_n - x\| \longrightarrow 0$. $x = \sum_{j=1}^{\infty} \alpha_j u_j$

$\Rightarrow$ $\langle x, u_j \rangle = \alpha_i$ $\therefore x = \sum_{j=1}^{\infty} \langle x, u_j \rangle u_j$. $\langle x, u_j \rangle = \alpha_j \neq 0 \ \forall j$

Suppose E is a basis.

$\Rightarrow \exists \beta_1, \ldots, \beta_k \in \mathbb{K}$ & $v_1, \ldots v_k \in E \ni x = \sum_{i=1}^{k} \beta_i v_i$

$\therefore \sum_{j=1}^{\infty} \alpha_j u_j = \sum_{i=1}^{k} \beta_i v_i$

Let $n \in \mathbb{N}$ be $\ni u_n \notin \{v_1, \ldots, v_k\}$ taking innerprod with

$\Rightarrow \alpha_n = 0$ contradicts the assumption that $\alpha_n \neq 0$. $\forall n$.

* Every separable infinite dim. Hilbert space is linearly isometric with ℓ^2.

$x = \sum_{j=1}^{\infty} \langle x, u_j \rangle u_j \xmapsto{T} (\langle x, u_1 \rangle, \langle x, u_2 \rangle, \ldots) \in \ell^2$

Parseval's $\Rightarrow$ $\sum |\langle x, u_j \rangle|^2 = \|x\|^2$.

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$\rightarrow \quad L^2[0,2\pi] \longrightarrow \chi$

$x = \sum \hat{x}(n) \frac{e^{int}}{\sqrt{2\pi}} \longmapsto (\hat{x}(1), \hat{x}(2), \dots)$

$\hat{x}(n) = \langle x, u_n \rangle \quad u_n(t) = \frac{e^{int}}{\sqrt{2\pi}}$

Linear operators.

Let X & Y be nls. $A: X \longrightarrow Y$ be a linear operator.

Thm A is continuous at 0

$\Leftrightarrow \exists\ c>0 \ni \|Ax\| \leq c\|x\| \ \forall x \in X.$

$\Rightarrow \|A(x-u)\| \leq c\|x-u\|$ → uniform continuity → Lipschitz

• Suppose A is continuous at 0.

18/

* TFAE,

1) A is continuous at 0
2) $\exists\ c>0 \ni \|Ax\| \leq c\|x\| \ \forall x \in X$
3) Image of every bdd set in X under A is bdd in Y.
4) $\{Ax : \|x\| \leq 1\}$ bdd
5) $\{Ax : \|x\| = 1\}$ bdd
6) $\{\|Ax\| : \|x\| < 1\}$ bdd.
7) $\{Ax : \|x+x_0\| < r\}$ bdd for some x_0.

$\mathcal{B}(X,Y)$ set of all bdd linear operators from X to Y

→ Linear space.

$A \in \mathcal{B}(X,Y)$ $\quad \|A\| = \sup_{\|x\| \le 1} \|Ax\| < \infty.$

$A \mapsto \|A\|$ is a norm on $\mathcal{B}(X,Y)$.

Y is a Banach space.

$\Rightarrow$ $\mathcal{B}(X,Y)$ Banach space.

In particular, $\mathcal{B}(X, \mathbb{K})$ is a Banach space $\forall$ nls X.

$X' = \mathcal{B}(X, \mathbb{K})$ dual space of X.

- Let X be f.d. $\{u_1, \dots, u_n\}$ basis.

$f_j : X \longrightarrow \mathbb{K}$ by $f_j\left(\sum_{i=1}^{n} a_i u_i\right) = \alpha_j$.

$|f_j(x)| = |\alpha_j| \le \|x\|_E \le C\|x\|$.

$f_j(u_j) = \delta_{ij}$ $\quad f_j \in X'$

$\Rightarrow \sum \alpha_j f_j = 0 \quad \Rightarrow \sum \alpha_j f_j(u_j) = 0 \Rightarrow a_i = 0.$

$\{f_1, \dots, f_n\}$ is a basis of X'.

$\dim X' = \dim X$.

* Corollary of HBET

$x \neq 0 \Rightarrow \exists f \in X' \ni \|f\| = 1$ & $f(x) = \|x\|$.

$\Rightarrow \{u_1, \dots, u_n\}$ L.I, $\exists \{f_1, f_2, \dots, f_n\} \subseteq X'$.

$\ni f_j(u_j) = \delta_{ij}$

* X, X', X''.

$X \longrightarrow X''$

$x \longmapsto \phi_x$ $\quad$, $\phi_x : X' \longrightarrow \mathbb{K} \ni \phi_x(f) = f(x)$

ϕ_x is linear.

$|\phi_x(f)| = |f(x)| \leq \|f\| \, \|x\|$. $\|f\| \leq 1$ by def. of norm.

$\Rightarrow \phi_x \in X''$ & $\|\phi x\| \leq \|x\|$

$0 \neq x \in X \ni f \ni f(x) = \|x\|$ & $\|f\| = 1$.

$\|x\| = \|\phi x\|$

ϕ_x is a linear isometry.

$x \longmapsto \phi_x$ need not be onto.

(eg: if X is not complete then J is not onto)

Def. A nls X is said to be reflexive if J is onto.

- $C[a,b]$ is not reflexive wrt any norm.
- Every f.d. space is reflexive.
- Every Hilbert space is reflexive.
- ℓ^p is reflexive iff $1 < p < \infty$.
- $L^p[a,b]$ is reflexive iff $1 < p < \infty$.

Ex ℓ^∞ is not a separable space.

L^∞ is not separable.

- C_{00} is not reflexive

Whether $C[a,b]$ has a denumerable basis.

5K. Distributions and Sobolev spaces.

shocks ?

Measure — Distribution.

→ Test f^n: $f : \mathbb{R} \longrightarrow \mathbb{R}$.

$$f(x) = \begin{cases} \exp(-x^{-2}) & \text{if } x > 0 \\ 0 & \text{if } x \leq 0. \end{cases}$$

Then $f \in C^\infty$.

$$\rho(x) = \begin{cases} e^{\left(\frac{-a^2}{(a^2 - x^2)}\right)} & |x| < a \\ 0 & \text{if } |x| \geq a. \end{cases}$$

→ Mollifiers. $x \in \mathbb{R}^N$,

$$\rho_\varepsilon(x) = \begin{cases} k\varepsilon^{-N} \exp\left(-\varepsilon^2/(\varepsilon^2 - |x|^2)\right) & |x| < \varepsilon \\ 0 & |x| \geq \varepsilon \end{cases}$$

where $k^{-1} = \int_{|x| \leq 1} e^{\left(\frac{-1}{(1 - |x|^2)}\right)} dx$

The family of f^ns $\{\rho_\varepsilon\}_{\varepsilon > 0}$ — family of mollifiers.

→ Partition of unity.

→ Cut off f^ns.

[In TVS sequential conty. exists but f^n may not be continuous]

→ $D(\Omega)$ is topological v.s. (not nls)

How do we use the definition of convergence in $D(\Omega)$?

→ Distributions.

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Not all f's are distributions — Generalized f's ? biggest class of f's ?
Is $L^1_{loc} \to \{f's\}$.

→ f is locally integrable ⇔ ∀ compact subset $K \subset \Omega$

$$\int_K |f| \, d\mu < \infty .$$

$$f(x) = \frac{1}{|x|} \in L^1_{loc}(\mathbb{R}^2).$$

$$f(x) = \frac{1}{|x|^2} \in L^1_{loc}(\mathbb{R}^3)$$

g: $f \in L^1_{loc}(\Omega)$, $\Omega \subset \mathbb{R}^N$ define $T_f : D(\Omega) \to \mathbb{R}$ by

$$T_f(\phi) = \int_\Omega f \phi \, dx .$$

$$|T_f(\phi)| \le \|\phi\|_\infty \int_K |f| \, dx .$$

$K = \text{supp}\, \phi$

→ Dirac distribution.

$\delta_x(\phi) = \phi(x)$ $\quad \phi \in D(\mathbb{R}^N)$. graph of δ_x.

→ μ be either a complex Borel measure or a +ve measure on $\mathbb{R}^N$ which is finite on compact sets.

$$T_\mu(\phi) := \int_{\mathbb{R}^N} \phi \, d\mu \qquad \phi \in D(\mathbb{R}^N)$$

→ Dipole distribution. (doublet)

1) $\phi \longmapsto \phi'(0)$ on $D(\mathbb{R})$

2) $\phi \longmapsto \phi^{(m)}(0)$ defines distributions.

→ order of a distribution.

All the eg: of distributions except doublet distribution, is of order 0.
its of order 1.

→ $D^\alpha = \dfrac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_N^{\alpha_N}}$ multiindex.

$N=3$ $\alpha = (2,0,5)$ $D^\alpha = \dfrac{\partial^7}{\partial x_1^2 \, \partial x_3^5}$.

→ Derivatives of distributions

→ infinitely differentiable.

→ Derivatives are distribut^{n}s.

Derivative of distribution from measure?

If we have some f^n, how to check whether that gives a distribut^n?

→ Derivative of any distribution T, we define $T'(\phi) := -T(\phi')$

$\alpha \to$ multi index $D^\alpha T(\phi) = (-1)^{|\alpha|} T(D^\alpha \phi)$

$\delta'(\phi) = -\delta(\phi') = -\phi'(0)$. upto a sign, doublet distribut^n.

→ Heaviside f^n on $\mathbb{R}$, $H(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0. \end{cases}$

$T_H'(\phi) = \delta(\phi)$

$T_H' \neq T_{H'}$
$\overset{\shortparallel}{\delta}$ $\qquad$ $\overset{\shortparallel}{0}$-distribution. ($\because$ $\exists$ classical derivative a.e $= 0$)

If $\int f'\phi \, dx = -\int f \phi' \, dx$ then $T_f' = T_{f'}$.

For absolutely continuous f^ns. (ie, satisfied fundamental thm of calculus ie, $f(a) - f(b) = \int_a^b f'(t)\,dt$)

→ Cantor f^n derivative exists on a.e. $= 0$.

Cantor f^n is not absolutely continuous.

MTN

∴ Let X be a f.d. nls $\{u_1, u_2, \dots, u_n\}$ basis of X.
Let Y be a nls. Let $A : X \longrightarrow Y$ be a linear map.
Claim $A \in \mathcal{B}(X, Y)$.

- Let $X = C[a,b]$ with $\|\cdot\|_\infty$

  Let $t_0 \in [a,b]$

  $$|f(x)| := |x(t_0)| \leq \|x\|_\infty .$$

- Consider $\|\cdot\|_1$ on $C[a,b]$ $\|x\|_1 = \int_a^b |x(t)|\,dt$

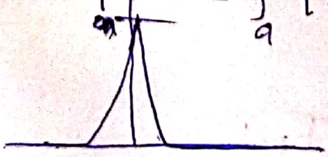
  Is $x \longmapsto x(t_0)$ continuous?

  $f(x_0) = x(t_0)$ $x \in C[a,b]$

  $X = C[a,b]$ with $\|\cdot\|_1$ $f : X \longrightarrow \mathbb{K}$.

  Does $\exists$ $c > 0$ $\ni$ $|x(t_0)| \leq c \int_a^b |x(t)|\,dt$ $\forall x \in C[a,b]$?

  No.



f is unbdd.

$X = C[0,1], \|\cdot\|_2$

$Y = X$

$A : X \longrightarrow Y.\ (Ax)(t) = f(x)t.$ Range depends on f ?

$\|Ax\|_2 = |f(x)| \frac{1}{2}.$ $\int_0^1 f(x)t\,dt$

$R(A) = \text{span}\{t\}.$

A is not bdd.

Ex Let $X_p = C[a,b], \|\cdot\|_p$ $1 \le p < \infty$.

$f(x) := x(t_0),\ f : X \longrightarrow \mathbb{K}$ linear f^n al.

S.T. f is continuous iff $p = \infty$.

* A linear op. $A : X \longrightarrow Y$ is said to be bdd below if $\exists\ \gamma > 0 \ni \|Ax\| \geq \gamma \|x\|\ \forall x \in X$

$\Leftrightarrow \|Ax\| \geq \gamma\ \forall x \in S = \{u \in X : \|u\| = 1\}$

$\Rightarrow$ A is 1-1. $\exists\ A^{-1} : R(A) \longrightarrow X$ is a linear operator.

Is A^{-1} bdd ?

$\|A^{-1}y\| \le M\|y\|\ \forall y \in R(A) \Leftrightarrow \|x\| \le M\|Ax\|\ \forall x$

$M = 1/\gamma.$

- $A : X \longrightarrow Y$ bdd below. Then $\forall y \in R(A),\ \exists!\ x \in X \ni$

$Ax = y$

and if $y_n \in R(A) \ni y_n \longrightarrow y$ and if $A^{-1}y_n = x_n$ then $x_n \longrightarrow x = A^{-1}y.$

Thm Suppose $A : X \longrightarrow Y$ is a bdd op. which is bdd below. If X is a Banach space, then $R(A)$ is closed.

We've $\|Ax\| \geq \gamma \|x\|$

$y_n = Ax_n \quad \| Ax_n - Ax_m \| \geq \gamma \| x_n - x_m \|$

$\downarrow$

y

Thm Let $A : X \longrightarrow X$ be a bdd linear operator on a ~~Banach~~ Hilbert space X $\ni$ $\exists \gamma > 0 \ni$

$$| \langle Ax, x \rangle | \geq \gamma \|x\|^2 \quad \forall x \in X.$$

Then A is bijective and $A^{-1} : X \longrightarrow X$ is continuous.

Applying Cauchy Schwartz we get $\|Ax\| \geq \gamma \|x\|$.

Conclusion : $\forall y \in X, \exists ! \, x \in X \ni Ax = y$ and A^{-1} continuous.

Projection Theorem

If X ; Hilbert space and X_0 is a closed subspace then $X = X_0 + X_0^{\perp}$. & $X_0^{\perp\perp} = X_0$.

Let $x \in X$.

X_0 is closed in $X \rightarrow X_0$ is Hilbert space.

Let E_0 be an onb of X_0.

Take $y = \sum_{u \in E_0} \langle x, u \rangle u \in X_0$

$$x = \underset{\in X_0}{y} + (x - y)$$

$\langle x-y, v\rangle = \langle x, v\rangle - \langle y, v\rangle$
$= 0 \qquad \forall\, v \in E_0.$

$\Rightarrow (x-y) \in X_0^{\perp}$

(Fourier expansion)

[Ex. For any $S \subseteq$ an ips X, $S^{\perp}$ is a closed subspace]

Let X be an ips ; $u \in X$.

Define $f_u : X \longrightarrow K$ by $f_u(x) = \langle x, u\rangle \quad x \in X.$

$f_u \in X'$ & $\|f_u\| \leq \|u\|$.

& $\|u\|^2 = \langle u, u\rangle = |f_u(u)| \leq \|f_u\|\,\|u\|$

$\Rightarrow \|u\| \leq \|f_u\|$, $\|f_u\| = \|u\|$.

RRT

If X is a Hilbert space and $f \in X'$ then $\exists\, u \in X \ni f(x) = \langle x, u\rangle \ \forall x \in X$ and $\|f\| = \|u\|$.

Pf/ Three if $f = 0$ (Can take $u = 0$).

If $f \neq 0$.

$\exists\, x_0 \in X \ni f(x_0) \neq 0$ let $x_0 \in N(f)^{\perp}$.

For any $x \in X$

$$x = \underbrace{\left(x - \frac{f(x)}{f(x_0)} x_0\right)}_{N(f)} + \underbrace{\frac{f(x)}{f(x_0)} x_0}_{\text{span}\{x_0\} = N(f)^{\perp}}.$$

* Let X be a linear space. Then a map $\phi : X \times X \to \mathbb{K}$ is said to be a sesquilinear f^n^al (bilinear if $\mathbb{K} = \mathbb{R}$)

if $(x, y) \longmapsto \phi(x, y)$

is linear wrt first variable & conjugate lin. wrt 2^nd^ variable.

On $X \times X$ $\quad \|(x,y)\|_{X \times X} = \|x\|_X + \|y\|_X$.

* ϕ is continuous $\iff \exists\, c > 0 \ni$

$$|\phi(x,y)| \le c\|x\| \, \|y\| \quad \forall (x,y) \in X \times X.$$

or $\quad \forall (x_n, y_n)$ in $X \times X$

$$(x_n, y_n) \longrightarrow (x,y) \Rightarrow \phi(x_n, y_n) \longrightarrow \phi(x,y).$$

eg: X is an ips.

$$\phi(x,y) = \langle x, y \rangle$$

$A : X \to X$ is a lin. op.

$$\phi(x,y) = \langle Ax, y \rangle$$

A bdd op. $\Rightarrow \phi$ is conti.

<u>Thm</u> Let X be a Hilbert space and ϕ be a continuous Sesquilinear f^n^al on X.

Then $\exists$ a continuous lin. op. $B : X \to X \ni$

$$\phi(x,y) = \langle x, By \rangle \quad \forall (x,y) \in X \times X$$

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(Lax-Milgram Thm)

Pf: For each y, $x \longmapsto \phi(x,y)$ conts. linear f^n als

$\Rightarrow$ by RRT $\exists!$ $u_y \in X \ni$

Define $\phi(x,y) = \langle x, u_y \rangle$

$B : Y \longrightarrow Y$ as $By = u_y$.

$A : X \longrightarrow X$ bdd lin. op

SK.

$\frac{dT}{dx}$ is zero distribution $\Rightarrow T(\phi) = c\int_{\mathbb{R}} \phi \, dx$.

ie, $0 = \frac{dT}{dx}(\phi) = -T(\phi')$.

→ Multiplication of distribut^n by smooth f^ns.

$(\psi T)'(\phi) = -(\psi T)\phi' = -T(\psi\phi')$

$\psi \in C^{\infty}(\mathbb{R})$, $T \in D'(\mathbb{R})$, $\phi \in D(\mathbb{R})$

→ $(\psi T)' = \psi' T + T' \psi$.

→ $D^{\alpha}(\psi T)$ → Leibniz formula.

→ ρ_ε → mollifiers, $\rho_\varepsilon \longrightarrow \delta$ as $\varepsilon \to 0$ in the sense of distributions.

→ Sobolev spaces.

$H^m \quad (u,v)_{m,\Omega} = \sum_{|\alpha| \le m} \int_\Omega D^\alpha u \, D^\alpha v \, dx.$

$u \longmapsto (u, u_{x_1}, \dots, u_{x_N})$

is an isometry from $W^{1,P}(\Omega)$ onto a subspace $(L^P(\Omega))^{N+1}$

→ Banach Alaglu

→ $W_0^{m,P}(\Omega) = W^{m,P}(\Omega)$ if $\Omega = \mathbb{R}^n$.

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MTN.

Backward Heat conduction Problem (BHCP).

An Ill-Posed problem.

~~How do we know error is there?~~

SG imkm

* $A : X \longrightarrow Y$. A has continuous inverse from its range iff A is bdd below.

* X & Y are Hilbert spaces. $A \in \mathcal{B}(X,Y)$.
Then A is a compact ~~bdd~~ operator iff ∃ (A_n) of finite rank bdd operators ∋
$\|A_n - A\| \longrightarrow 0$ as $n \to \infty$.

$u_t = c^2 u_{xx} \quad 0 < x < l, \; t > 0$

$u(x,0) = f_0$

$u(0,t) = 0 = u(x,t)$

$$u(x,t) = \sum e^{-c^2 n^2 \frac{n\pi}{l} t} \hat{f}(n) \sin \frac{n\pi}{l} t.$$

$$= \sum e^{-c^2 n^2 \frac{n\pi}{l} t} \langle f_0, \phi_n \rangle \phi_n.$$

→ Inverse problems

→ Piccard condition.

$$A\phi = \sum_{n=1}^{\infty} e^{-\lambda_n^2 (\tau - t_0)} \langle \phi, \phi_n \rangle \phi_n.$$

→ Truncated singular value representation method

→ Singular value decomposition.

SK

Convolutions:

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-t)\, g(t)\, dt.$$

→ $\rho_\varepsilon * u \longrightarrow u$ in $L^p(\mathbb{R}^N)$ for $u \in L^p(\mathbb{R}^N)$

→ uniform convergence & L^p convergence. on $L^p(\Omega)$

Ω - compact.

→ Continuous fns with compact supp is dense in $L^p(\mathbb{R}^n)$.

Theorem

For $1 \le p < \infty$, we have

$$W_0^{m,p}(\mathbb{R}^N) = W^{m,p}(\mathbb{R}^N)$$

change when $\Omega \subset \mathbb{R}^N$.

→ <u>Friedrich's theorem</u>.

let $1 \le p < \infty$. $\Omega \subset \mathbb{R}^N$ open set. Let $u \in W^{1,p}(\Omega)$.

Then $\exists$ a sequence $\{u_n\} \in D(\mathbb{R}^N)$ $\ni$

$u_n \longrightarrow u$ in $L^p(\Omega)$ & $\frac{\partial u_n}{\partial x_i} \longrightarrow \frac{\partial u}{\partial x_i} \in L^p(\Omega')$

for any relatively compact open set Ω' of Ω.

($\because$ convolution is defined only on $\mathbb{R}^N$)

→ Extension operators. (E.O) –

let $\Omega \subset \mathbb{R}^N$ be an open set. A bdd lin. op.

$P : W^{1,p}(\Omega) \longrightarrow W^{1,p}(\mathbb{R}^N)$ is an extension op.

if ~~the~~ $Pu|_{\Omega}$ is u $\forall u \in W^{1,p}(\Omega)$.

<u>Thm</u> If $\exists$ a P on $W^{1,p}(\Omega)$ then for any

$u \in W^{1,p}(\Omega)$ $\exists$ a $\{u_n\} \in D(\mathbb{R}^N)$ $\ni$

$u_n \longrightarrow u$ in $W^{1,p}(\Omega)$.

* Existence of E.O ?

Depends on Ω.

* If $\exists$ an extension op. on $W^{1,p}(\Omega)$, then

$C^\infty(\bar{\Omega})$ is dense in $W^{1,p}(\Omega)$.

→ chain rule in $W^{1,p}(\Omega)$

whether $\exists$ these things when $\Omega = \mathbb{R}^N$?

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→ stampacchia : finite number of points.

4^{th} order elliptic - weak formulations are in $W^{2,p}$

RKG

Controllability of Dynamical Systems.

(Dynamical system, coupled biharmonic)

Unforced eqn. $\frac{dx}{dt} = -2x$ $x(0) = 3$.

Control $\frac{dx}{dt} = -2x + \sin t$ $x(0) = 3$.

The system with a forcing term is called control system.

$$\frac{dx}{dt} = G(t, x(t), u(t))$$

$$x(0) = x_0.$$

→ Trajectory controllability path $\tilde{x}(t)$, $\tilde{x}(t_0) = x_0$ $\tilde{x}(t_1) = x_1$.

→ Complete controllability. $x(t_0) = x_0$, $x(t_1) = x_1$.

→ Kalman

→ Reachable set.

→ linearized system

$G(t, x(t), u(t)) \approx A(t)x(t) + B(t)u(t)$.

$\dot{x}(t) = A(t)x(t) + B(t)u(t)$, $x(t_0) = x_0$.

$\dot{x}(t) = A(t)\, x(t)$ — ① n-Dim. e^{At}.

$\phi(t) = [\phi_1(t) \quad \phi_2(t) \quad \cdots \quad \phi_n(t)] \to$ fundamental matrix.

$\{\phi_i\}$ linearly indep. solns of ①

Fundamental matrix soln.

linear systems $\Rightarrow$ Fundamental matrix ~~sol~~ at particular (fixed) t_0 is non-singular.

→ Transition matrix.

→ Peano-Baker series

$$\phi(t, t_0) = I + \int_{t_0}^{t} A(\tau_1)\, d\tau_1 + \int_{t_0}^{t} A(\tau_1) \int_{t_0}^{\tau_1} A(\tau_2)\, d\tau_2\, d\tau_1$$

→ Adjoint of the control operator.

→ Controllability Grammian matrix.

→ system is controllable if Grammian is non-zero.

$(CC^*)^{-1}$ generalized inverse.

SK

Poincare's Inequality

for any Ω $\exists$ P, E.O. from $W_0^{1,P}(\Omega)$ to $W^{1,P}(\mathbb{R}^N)$

let $\Omega \subset \mathbb{R}^N$ be a bdd open set: $1 \leq P < \infty$.

Then $\exists$ $C = C(P, \Omega) > 0$ $\ni$

$$|u|_{0,p,\Omega} \leq C|u|_{1,p,\Omega}$$

$\forall\ u \in W_0^{1,p}(\Omega)$.

$\int_{\partial\Omega} \Delta u\, dx = 0$.

$\Rightarrow \nabla u = 0$ on $\partial\Omega$

??

This inequality shows $W_0^{1,p}(\Omega) \neq W^{1,p}(\Omega)$ Ω is bdd.

eg: $u \equiv 1$ on Ω $|u|_{1,p,\Omega} = 0$ but $|u|_{0,p,\Omega} > 0$.

C depends on diameter of Ω.

$p=2$ C ~~depe~~ is connected to the first eig. value of Δ with homog. Dirichlet problem.

* $\Omega = (0,1)$, $-u'' = \lambda u$ $u = 0$ on $\partial\Omega$.

* <u>Imbedding Theorems</u>

$u \in W^{1,p}(\Omega) \Rightarrow u \in L^p(\Omega)$.

- $W^{1,p}(\mathbb{R}^N) \hookrightarrow L^{p^*}(\mathbb{R}^N)$ $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{N}$.

- let $1 \leq p < N$. Then we've the continuous inclusions

$W^{1,p}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ $\forall\ q \in [p, p^*]$

- $a^\alpha b^{1-\alpha} \leq \alpha a + (1-\alpha) b$. $a, b > 0$.

Let $p = N$. Then we've the continuous inclusion

$W^{1,N}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$

eg: $\Omega = B(0; 1/2) \subset \mathbb{R}^2$.

$$u(x) = \log \log \frac{2}{|x|} \qquad x \neq 0.$$

Then $u \in H^1(\Omega)$. ie, $p = 2 = N$, but clearly

Absolutely continuous f $\Rightarrow$ classical derivative = distribution derivative

$\rightarrow$ $u \in W^{2,p}(\mathbb{R}^N)$

$$p^{**} = \frac{1}{p^*} - \frac{1}{N} = \frac{1}{p} - \frac{2}{N}.$$

- $W^{m,p}(\mathbb{R}^N) \hookrightarrow C^k(\mathbb{R}^N)$ for $m > \frac{N}{p}$.

  Holder space.

Compactness theorems

bbd $\longrightarrow$ relatively compact.

Weakly convergent sequence $\longrightarrow$ norm convergent sequence

- $C^1[0,1] \hookrightarrow C[0,1]$

$\|f\|_1 = \max\{\|f\|_\infty, \|f'\|_\infty\}$ $\quad \|f\| = \max\{|f(x)|\}$

let $B \subseteq C^1[0,1]$ is a unit ball

$f \in B \Rightarrow \|f\|_1 \leq 1 \Rightarrow \|f\|_\infty \leq 1$

by MVT $\quad |f(x) - f(y)| \leq \|f'\|_\infty |x-y| = |x-y|$

equi continuous, uniformly bdd f's

· Arzela - Ascoli thm

$W^{1,p}(\Omega) \hookrightarrow C(\bar{\Omega})$ is compact by Arzela-Ascoli thm.

→ Theorem (Rellich – Kondrasov)

→ Frechet - Kolmogorov theorem.

→ We don't have compactness if Ω is unbdd.

eg: $\Omega = \mathbb{R}$. $I = (0,1)$ let $I_j = (j, j+1)$ $\forall j \in \mathbb{Z}$

let f be a C^1 fn, supported in I. Define

$$f_j(x) = f(x-j)$$

19/12/19

SK

RRT $H \sim H^*$

$H^m(\mathbb{R}^N)$ — Fourier transform.

Trace theory

BVP.

$u \in W^{1,p}$.

$\gamma_0 : H^1(\mathbb{R}^N_+) \longrightarrow L^2(\mathbb{R}^{N-1})$

$$\left.\begin{aligned} -\Delta u &= f \text{ in } \Omega \\ u &= g \text{ on } \partial\Omega \end{aligned}\right\} \quad \begin{aligned} &u \in H^1(\\ &\gamma_0(u) = g. \end{aligned}$$

• restrictions of f^n in $\mathcal{D}(\mathbb{R}^N)$ are dense in $H^1(\mathbb{R}^N_+)$

convolution of cut off.

range $(\gamma_0) = H^{1/2}(\mathbb{R}^{N-1})$ kernel $= H^1_0(\mathbb{R}^N_+)$

$\Omega \to \mathbb{R}^N_+$.

Green's thm (Corollary of trace theorem)

Thm. Let $1 < p < \infty$. $I = (0,1) \subset \mathbb{R}$. Then

$$W^{1,p}_0(I) = \{ u \in W^{1,p}(I) \mid u(0) = u(1) = 0 \}$$

$u, v \in H^1(\Omega)$.

$$\int u \frac{\partial v}{\partial x_i} = -\int_\Omega \frac{\partial u}{\partial x_i} v + \int_{\partial\Omega} \gamma_0(u)\, \gamma_0(v)\, \nu_i$$

Let $u \equiv 1$, $v = (v_1, \dots, v_n)$

$$\int \frac{\partial v_i}{\partial x_i} = \int_{\partial\Omega} \gamma_0(v_i)\, \nu_i$$

Sum,

$$\int \operatorname{div} v = \int_{\partial\Omega} v \cdot \nu \quad \to \text{Divergence theorem.}$$

$$\int \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} = -\int \frac{\partial^2 u}{\partial x_i} v + \int_{\partial} \frac{\partial u}{\partial x_i} v \cdot \nu_i$$

Summat^n,

$$\int_\Omega \nabla u \cdot \nabla v = -\int \Delta u\, v + \int_{\partial\Omega} \frac{\partial u}{\partial \nu} \cdot v$$

→ Green's identity.

A·A

First order PDE.

- How to solve linear PDE.
- Non linear ODE - Picard's theorem.
- Vector fields → Integral curves.
- First order PDE → Derivation of characteristic V.F.
- Local existence.

- Global existence.
- conservation laws → weak soln.

Riemann integral

$$\int_a^b f(x)\,dx, \qquad \int_a^b \frac{df}{dx}\,dx = f(b) - f(a)$$

1) $\frac{du}{dx} = 0 \quad (a,b) \quad ; u = c.$

2) $\frac{du}{dx} = f(x) \quad u(x_0) = \alpha \quad ; \int_{x_0}^{y} \frac{du}{dx}\,dx = \int_{x_0}^{y} f(x)\,dx.$

$$u(x) = u(x_0) + \int_{x_0}^{y} f(x)\,dx.$$

3) $\frac{du}{dx} = P(x)u(x) + Q(x), \quad u(x_0) = \alpha$

using I.F. $\quad$ I.F $= R(x)$.

$$e^{-R(x)}u(x) = e^{-R(x)}u(x_0) + \int_{x_0}^{x} e^{-R(t)}Q(t)\,dt.$$

4) $\frac{du}{dt} - au = Q(x)$ when $P(x) = a$ in (3).

$u(0) = \alpha$.

$$u(x) = e^{R(x)}\alpha + \int_{x_0}^{x} e^{(R(x)-R(t))}Q(t)\,dt.$$

ie, $u(x) = e^{ax}\alpha + \int_{x_0}^{x} e^{a(x-t)} Q(t)\,dt$.

5) $u = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$ $A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ linear map

$$\frac{du}{dt} = Au + Q(t).$$

$$u(0) = u_0 = \begin{pmatrix} u_{01} \\ \vdots \\ u_{0n} \end{pmatrix}.$$

$$\begin{cases} \dfrac{d^n u}{dt^n} + a_{n-1}\dfrac{d^{n-1}u}{dt^{n-1}} + \dots + a_0 u = f \\ u(0) = \alpha_0,\ u'(0) = \alpha_1,\ \dots\ u^{(n-1)}(0) = \alpha_n. \end{cases}$$

[$$\begin{cases} \dfrac{d^2u}{dt^2} + a_1\dfrac{du}{dt} + a_0 u = 0 \\ u(0) = \alpha_0,\ u'(0) = \alpha_1 \\ m^2 + a_1 m + a_0 = 0 \qquad (m - m_1)(m - m_2) = 0. \end{cases}$$

$$\left(\frac{d}{dt} - m_1\right)\left(\frac{d}{dt} - m_2\right)u = \frac{d^2u}{dt^2} + a_1\frac{du}{dt} + a_0 u = 0.$$

$\left(\frac{d}{dt} - m\right)(u) = 0$ solve by (4)]

let $u_0 = u$ $\qquad \frac{du_0}{dt} = u_1$.

$u_1 = \frac{du}{dt}$

$u_2 = \frac{d^2u}{dt^2}$

$\vdots$

$u_n = \frac{d^n u}{dt^n}$

$$\frac{d^2u}{dt^2} + a_1\frac{du}{dt} + a_0 u = f.$$

$$u_0 = u$$
$$u_1 = \frac{du}{dt}$$

$$\frac{d}{dt}\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \begin{pmatrix} u_1 \\ \frac{d^2u}{dt^2} \end{pmatrix} = \begin{pmatrix} u_1 \\ -a_1u_1 - a_0u_0 + f \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -a_0 & -a_1 \end{pmatrix}}_{A}\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ f \end{pmatrix}}_{B}$$

6) $$\begin{cases} \frac{dU}{dt} = AU + B. & U = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}. \\ U(0) = \begin{pmatrix} u_0(0) \\ u_1(0) \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \end{cases}$$

$$U(t) = e^{At}U(0) + \int_0^t e^{Ax - As} B(s)\,ds.$$

$$e^a = \sum_{n=0}^{\infty} \frac{a^n}{n!} \quad \text{define} \quad e^{At} = \sum_{n=0}^{\infty} \frac{(At)^n}{n!} = \sum_{n=0}^{\infty} \frac{t^n A^n}{n!}$$

$$\|A\| = \sup_{\substack{x,y \in \mathbb{R}^n \\ \|x\| \le 1, \|y\| \le 1}} \langle Ax, y \rangle.$$

$$\|A_1A_2\| \le \|A_1\|\,\|A_2\|.$$
$$\|A_1 + A_2\| \le \|A_1\| + \|A_2\|.$$

$$z(t) = e^{tA}U(0) = \sum_{n=0}^{\infty} \frac{t^n A^n U(0)}{n!}$$

$$\frac{d}{dt}e^{tA} = \sum_{n=0}^{\infty} \frac{t^{n-1}A^n}{(n-1)!} = A\sum_{n=0}^{\infty} \frac{t^{n-1}A^{n-1}}{(n-1)!}$$

$$= A\sum_{n=0}^{\infty} \frac{t^n A^n}{n!} = Ae^{tA}$$

$$\frac{dz}{dt} = Az(t)$$

Now consider,

7) $\dfrac{dU}{dt} = A(t)U + B(t)$.

$U(t) = \begin{pmatrix} u_1(t) \\ \vdots \\ u_n(t) \end{pmatrix}$ $\quad A(t)_{n\times n} \quad B(t) = \begin{pmatrix} b_1(t) \\ \vdots \\ b_n(t) \end{pmatrix}$

$U(0) = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$.

$$\begin{cases} \dfrac{dU}{dt} = F(U, t) \\ U(0) = U_0 . \end{cases}$$

$$\int_0^t \frac{dU}{ds}\, ds = \begin{pmatrix} \int_0^t \frac{du_1}{ds}\, ds \\ \vdots \\ \int_0^t \frac{du_n}{ds}\, ds \end{pmatrix} = \begin{pmatrix} u_1(t) - u_1(0) \\ \vdots \\ u_n(t) - u_n(0) \end{pmatrix}$$

$$\frac{du_1}{dt} = F_1(u_1, \dots, u_n, t)$$

$$\vdots$$

$$\frac{du_n}{dt} = F_n(u_1, \dots, u_n, t)$$

$$\int_0^t \frac{dU}{ds}\, ds = \begin{pmatrix} \int_0^t F_1(U, s)\, ds \\ \vdots \\ \int_0^t F_n(U, s)\, ds \end{pmatrix}.$$

$$\Rightarrow \boxed{U(t) = U(0) + \int_0^t F(U(s), s)\, ds .}$$

Solving the integral eqn.

Solve $F: X \longrightarrow Y \ni F(x) = y$.

Now let $Y = X$. Solve $F(x) - y = 0$.

$$\underbrace{F(x) - y + x}_{G(x)} = x.$$

$$G(x) = x.$$

$\Rightarrow x$ is a fixed pt.

Contraction mapping thm

$X \to$ complete metric space.

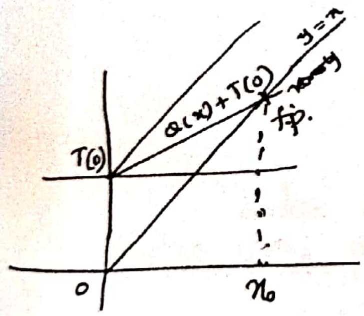
$T: X \longrightarrow X$ a map. $d(Tx, Ty) \leq \alpha\, d(x,y) \;\forall (x,y)$

$0 \leq \alpha < 1$.

then $\exists !\; x_0 \in X \ni Tx_0 = x_0$

$T: \mathbb{R} \longrightarrow \mathbb{R}$. $|T(x) - T(0)| \leq \alpha |x|$ $\quad x > 0 \quad T(0) > 0$.

$$-\alpha x + T(0) \leq T(x) \leq T(0) + \alpha x.$$



Assumptions.

1) $\Omega \subseteq \mathbb{R}^n$ be open.

2) $F: \Omega \times \mathbb{R} \longrightarrow \mathbb{R}^n$ be a continuous map. $\quad U \subset \mathbb{R}^n$ is open.

3) F is locally Lipschitz in U.

for a compact set

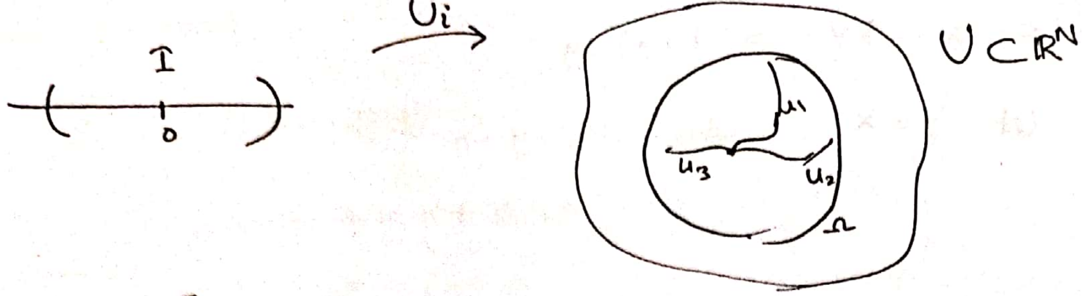
$|F(u_1; t) - F(u_2, t)| \leq M(K)\, |u_1 - u_2|$ $\quad \forall\, u_1, u_2 \in K \quad K \to$ compact

$\forall\, t \in$ Compact set in $\mathbb{R}$.

$M(K) \geq 0$.

Eg: $F(U, t) = A(t)U + B(t)$

$|F(U_1, t) - F(U_2, t)| = |A(t)(U_1 - U_2)| \leq \|A(t)\|\, |U_1 - U_2|$ $\quad \forall\, t \in L$



$X = \{ u : I \to \Omega \ : u(0) = \alpha \}$

$\|u_1 - u_2\| = \sup_{t \in I} |u_1(t) - u_2(t)|.$

$T : X \longrightarrow X.$

$T(u)(t) := a + \int_0^t F(u(s), s)\, ds.$

$$|T(u_1)(t) - T(u_2)(t)| \leq \int_0^t |F(u_1(s), s) - F(u_2(s), s))|\, ds.$$

$$\leq c\|u_1 - u_2\| t.$$

PSD

Laplace and Poisson Equations.

ODE / PDE

For a linear nth order ODE, $\exists$ only n l.ind. solns.

Soln space is finite dim. soln curves, soln surfaces.

$y' = f(t, y)$

$y(t_0) = y_0.$

First order PDE

$\downarrow$

System of ODE.

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- Integration in several variables.
- Change of variables (in integration)
- Divergence theorem
- Gauss – Ostrogadskii formula.

$\mathbb{R}^n$: Surface (hyper surface)

$$g_1(u_1, \cdots, u_{n-1}), \cdots, g_n(u_1, \cdots, u_{n-1})$$

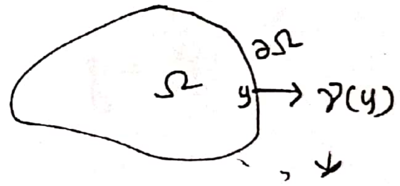
Condition: • $\left[\frac{\partial g_i}{\partial u_j}\right]_{n \times (n-1)}$ has max. rank.

• $F(x_1, \ldots, x_n) = 0$.

$B_r(x)$ be an open ball in $\mathbb{R}^n$.

$B_r(x) = \{ y : |x-y| < r \}$

$\partial B_r(x) = \{ y : |x-y| = r \}$



Outward unit normal to $\partial\Omega$ at y.

$$\int_{\Omega} (\operatorname{div} F)(x)\, dx = \int_{\partial\Omega} (F \cdot \nu(y))\, ds(y).$$

(To define unit normal ~~bdry of~~ smoothness of the bdry is needed)

<u>Laplace Eqn.</u>

$\Delta u = 0$ → then u is called a harmonic f^n for $n=2$.

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}.$$

∃ infinitely many complex f^ns, which are solns of

$$u_{xx} + u_{yy} = 0.$$

Ω is a bdd open set with smooth bdry $\partial\Omega$.

Dirichlet problem

Find $u \in C^2(\Omega) \cap C(\bar{\Omega}) \ni$

$C(\Omega)$ $C(\bar{\Omega})$
we cannot extend
$f \in C(\Omega)$ to $C(\bar{\Omega})$

$$\Delta u = 0 \text{ in } \Omega$$
$$u = g \text{ on } \partial\Omega$$

where g is a given continuous f^n.

Neumann problem

Find $u \in C^2(\Omega) \cap C^1(\bar{\Omega}) \ni$

$\nabla u \cdot \nu = \frac{\partial u}{\partial \nu}$

$$\Delta u = 0 \text{ in } \Omega$$
$$\frac{\partial u}{\partial \nu} = g \text{ on } \partial\Omega$$
(ν is the outward unit normal)

Poisson eqn

$$\Delta u = f.$$

Mean value property.

let u be harmonic in Ω. Then

$$u(x) = \frac{1}{\sigma_n R^{n-1}} \int_{\partial B_R(x)} u(y)\, \underbrace{ds(y)}_{\text{surface measure}}$$

σ_n = surface area of the unit sphere in $\mathbb{R}^n$.

$$\sigma_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}$$

for all $x \in \Omega$ and $R > 0$

$\ni \overline{B_R(x)} \subset \Omega$

Volume of unit ball $= \frac{\sigma_n}{n}$.

If $\Delta u \geq 0$, u is called subharmonic

$\Delta u \leq 0$, u is called superharmonic

u is subharmonic $\Rightarrow$ $$u(x) \leq \frac{1}{\sigma_n R^{n-1}} \int_{\partial B_R(x)} u(y)\, ds(y)$$

" superharmonic $\Rightarrow$ $u(x) \geq$ " .

<u>Proof</u>

For $0 < r < R$, let

$$h(r) = \frac{1}{\sigma_n r^{n-1}} \int_{\partial B_r(x)} u(y)\, ds(y).$$

let $|y-x| = r$, $y = x + r\xi$ $\quad |\xi| = 1.$ $\quad \frac{dy}{dr} = \xi$

then $$h(r) = \frac{1}{\sigma_n} \int_{\partial B_1(0)} u(x+r\xi)\, ds(\xi).$$

$$\frac{\partial h(r)}{\partial r} = \frac{1}{\sigma_n} \int_{\partial B_1(0)} \frac{\partial}{\partial r} u(x+r\xi)\, ds(\xi).$$

$u(x)$.

$$= \frac{1}{\sigma_n} \int_{\partial B_1(0)} \frac{\partial}{\partial \gamma} u(x+r\xi)\, ds(\xi)$$

$\frac{\partial u}{\partial r} = \nabla u \cdot \frac{x}{r}$

$x_1^2 + \cdots + x_n^2 = r^2.$

$\cancel{\Rightarrow \frac{1}{\sigma_n} \int_{\partial B_1(0)} \frac{\partial u}{\partial \gamma} (x+}$

ξ_i are outward unit normals.

$$\int_{B_r(x)} \Delta u(y)\, dy = \int_{\partial B_r(x)} \frac{\partial u}{\partial \gamma}(y)\, ds(y)$$

$$= r^{n-1} \int_{\partial B_1(0)} \frac{\partial u}{\partial \gamma}(x+r\xi)\, ds(\xi).$$

If $\Delta u = 0 \Rightarrow \frac{\partial h}{\partial r} = 0 \Rightarrow h \equiv c$.

$\left(\frac{\partial}{\partial x_i}(uv) = v\frac{\partial u}{\partial x_i} + u\frac{\partial v}{\partial x_i}\right)$

AA

$$\frac{dU}{dt} = F(U, t)$$

$U(0) = U_0$.

Picard's thm.

Weierstrass P-function.

$F : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}$.

Nonlinear eqn. $F(x, u, Du) = 0$.

1) $\frac{\partial u}{\partial t} = f(t)$ $\quad u(x) = u_0 + \int_0^x f(s)ds$.

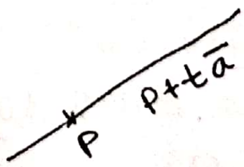
2) $\mathbb{R}^2$ $\quad x = (x_1, x_2)$

$\frac{\partial u}{\partial x_1} = 0 \quad u(x_1, x_2) = \phi(x_2)$

$\frac{\partial u}{\partial x_i} = 0 \quad u(x_1, \cdots, x_n) = \phi(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n)$

3) $Lu = \sum_{i=1}^{n} a_i \frac{\partial u}{\partial x_i}$, $Lu = 0$ $\quad a_i \in \mathbb{R}$.

$a = (a_1, a_2, \cdots, a_n) \neq 0$.



Consider $u(x_1, x_2, \cdots, x_n)$

$u|_g = g(t) = u(p+ta) = u(p_1+a_1t, p_2+a_2t, \ldots, p_n+a_nt)$

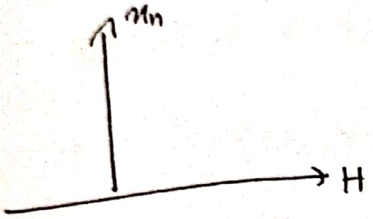
$$\frac{dg}{dt} = \sum_{i=1}^{n} a_i \frac{\partial u}{\partial x_i}(p+ta)$$

Assume u satisfies $Lu=0$

$$\frac{d}{dt}\, u(P+ta) = (Lu)(P+ta) = 0.$$

$$u(P+ta) = \text{Constant}. \qquad u(P+ta) = u(P).$$

$$H = \{x \in \mathbb{R}^n,\ \langle x, a\rangle = 0\}.$$



$$\frac{\partial u}{\partial x_n} = 0.$$

$$x = (x_1, \dots, x_{n-1}, 0)$$

$$Lu = \frac{\partial u}{\partial x_n} = 0$$

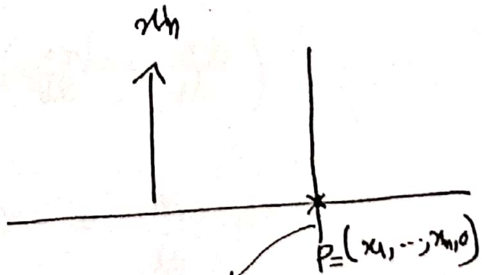
$$a = (0, 0, \dots, 0, 1)$$

$$u(x_1, x_2, \dots, x_n) = u(x_1, x_2, \dots, x_{n-1}, 0).$$

A vectorfield $X = \sum_{i=1}^{n} a_i(x) \frac{\partial}{\partial x_i}$ $\qquad a(x) = (a_1(x), \dots, a_n(x))$

let X be a vectorfield.

A curve in $\mathbb{R}^n$ passing thr. the point P in a smooth function $\begin{cases} \gamma : (-\varepsilon, \varepsilon) \longrightarrow \mathbb{R}^n \\ \gamma(0) = P \end{cases}$



is an integral curve of the v.f. $a(0,0,\dots 0,1)$

$\frac{d\gamma}{dt} \to$ tangent.

$$a(x) = (a_1(x), \dots, a_n(x))$$

Find $\gamma \ni a(\gamma(t)) = \frac{d\gamma}{dt}$.

If $\exists\ \gamma(t) \ni \frac{d\gamma(t)}{dt} = X$; the v.f. evaluated at t then γ is called integral curve.

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Aim Given a v.f. $X = \sum_{i=1}^{n} a_i(x) \frac{\partial}{\partial x_i}$ (or $a(x) = (a_1(x), \dots, a_n(x))$)

$P \in \mathbb{R}^n$.

Q. Does $\exists\ \gamma : (-\varepsilon, \varepsilon) \longrightarrow \mathbb{R}^n \ni$

i) $\gamma(0) = P$

ii) $\frac{d\gamma}{dt} = a(\gamma(t)) \quad \forall\ t \in (-\varepsilon, \varepsilon)$

Ex 1) $\mathbb{R}^2$ (x_1, x_2) $\quad x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} = X \quad a(x) = (x_1, x_2)$

Find $\gamma = (\gamma_1, \gamma_2) \ni$

$$\left(\frac{d\gamma_1}{dt} \quad \frac{d\gamma_2}{dt} \right) = a(\gamma(t)) = (\gamma_1(t) \ \ \gamma_2(t))$$

$$\frac{d\gamma_1}{dt} = \gamma_1(t) \qquad \frac{d\gamma_2}{dt} = \gamma_2(t).$$

$$\Rightarrow \gamma_1(t) = \gamma_1(0) e^{t}$$

$$\gamma_2(t) = \gamma_2(0) e^{t}.$$

$$P = (\gamma_1(0), \gamma_2(0))$$

$$\frac{\gamma_1(t)}{\gamma_2(t)} = \frac{\gamma_1(0)}{\gamma_2(0)} = c.$$

$$\Rightarrow x_1 = m x_2.$$

2) $(a_1(x), a_2(x)) = (-x_2, x_1)$

$$\left(\frac{d\gamma_1}{dt}, \frac{d\gamma_2}{dt} \right) = \frac{d\gamma}{dt} = (-\gamma_2, \gamma_1)$$

- $\gamma_1 \frac{d\gamma_1}{dt} = -\gamma_2$

$\gamma_2 \frac{d\gamma_2}{dt} = \gamma_1$

$$\frac{d}{dt}(x_1^2 + x_2^2) = 0.$$

$$\Rightarrow x_1^2 + x_2^2 = c.$$

Let $F(U) = (a_1(U), \dots, a_n(U))$ evaluate $a(U(t))$ for a particular t

$\gamma(t) = U(t)$

$x = U(t) \in \mathbb{R}^n$ for fixed t.

$$\begin{cases} \frac{dU}{dt} = F(U(t)) \\ U(0) = U_0 . \end{cases} \text{——} (1)$$

$$a(U(t)) = (a_1(U(t)), \cdots, a_n(U(t)))$$

$$\frac{dU}{dt} = a(U(t)) = F(U(t))$$

By Picard's theorem $\exists$ $U \ni U$ satisfies (1).

$\Rightarrow$ integral curve exists.

In PDE these integral curves are called characteristics.

(Dedekind's cut)

$$L u = \sum_{i=1}^{n} a_i(x) \frac{\partial u}{\partial x_i} = 0.$$

$$a(x) = (a_1(x), \cdots, a_n(x)) \rightarrow \text{v.f.}$$

$$\begin{cases} \frac{dx}{dt} = a(x(t)) \\ x(0) = x_0 . \end{cases}$$

By Picard's thm $\exists$ a soln $x = x(t, x_0)$

Suppose u is a soln $Lu = 0$.

<u>Restrict u on Γ</u>

Let $g(t) = u(x(t, x_0)) = u(x_1(t, x_0), \cdots, x_n(t, x_0))$

$$\frac{dg}{dt} = \sum_{i=1}^{n} \frac{\partial u}{\partial x_i} \frac{dx_i(t)}{dt} = \sum_{i=1}^{n} \frac{\partial u}{\partial x_i}(x(t, x_0))\, a_i(x(t, x_0))$$

$$= \sum_{i=1}^{n} \left(a_i \frac{\partial u}{\partial x_i}\right)(x(t, x_0)) = 0.$$

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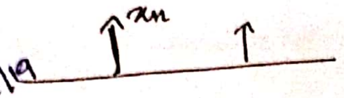

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AA

$$I \times \Omega' = (-T, T) \times \Omega'$$

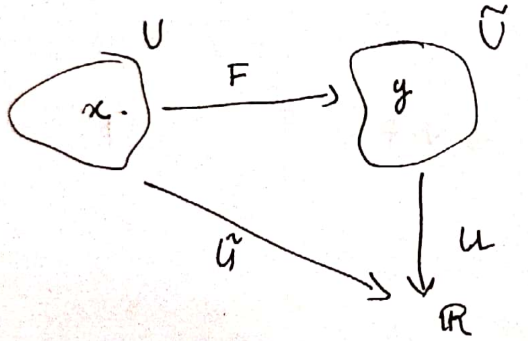
$$\phi : I \times \Omega' \longrightarrow \mathbb{R}^n .$$
$$(t, x') \longrightarrow \phi(t, x')$$

$$\phi(t, x') = (\phi_1(t, x'), \ldots, \phi_n(t, x'))$$

$$\left\{\begin{aligned} &\frac{d\phi}{dt}(t, x') = a_+ (\phi(t, x')) \\ &\frac{d\phi_1}{dt}(t, x') = a_1(\phi(t, x')) \\ &\vdots \\ &\frac{d\phi_n}{dt}(t, x') = a_n(\phi(t, x')) \\ &\phi(0, x) = (x', 0) \\ &\phi_1(0, x') = x_1 \\ &\vdots \\ &\phi_{n-1}(0, x') = x_{n-1} \\ &\phi_n(0, x') = 0 . \end{aligned}\right.$$

$$\phi'(0,0) = J_\phi = \begin{pmatrix} \frac{\partial \phi_1}{\partial t} & \frac{\partial \phi_1}{\partial x_1} & \cdots & \frac{\partial \phi_1}{\partial x_{n-1}} \\ \vdots & & & \\ \frac{\partial \phi_n}{\partial t} & - & - & \frac{\partial \phi_n}{\partial x_{n-1}} \end{pmatrix}$$

$$\phi'(0,0) = \begin{pmatrix} a_1(0) & 1 & 0 & \cdots & 0 \\ \vdots & 0 & 1 & \cdots & 0 \\ a_n(0) & 0 & \cdots & \cdots & 1 \\ & 0 & \cdots & \cdots & 0 \end{pmatrix} \qquad \det \phi'(0,0) = |a_n(0)| \neq 0 .$$



u

~~the~~ $\exists$ Diffeomorphisms by inverse mapping thm.

$$\frac{\partial g}{\partial t} = 0$$

$$g(t, x') = g(0, x')$$

$\rightarrow$ $Lu = \sum_{i=1}^{n} a_i(x) \frac{\partial}{\partial x_i} \qquad x \in \Omega \quad \rightarrow L(\alpha u_1 + \beta u_2) = \alpha(L(u_1)) + \beta L(u_2)$

$$L(u) = \left(\frac{\partial u}{\partial x_1}\right)^2 + \left(\frac{\partial u}{\partial x_2}\right)^2 = 1$$

$$F(x,u,p) = \sum_{i=1}^{n} a_i(x) p_i$$

$$F(x,u,p) = p_1^2 + p_2^2 - 1.$$

$$F : \underset{x}{\mathbb{R}^n} \times \underset{u}{\mathbb{R}} \times \underset{Du}{\mathbb{R}^n} \longrightarrow \mathbb{R} \qquad F(x,u,p) - \text{smooth}.$$

Aim To find u - smooth soln $F(x, u(x), Du(x)) = 0$.

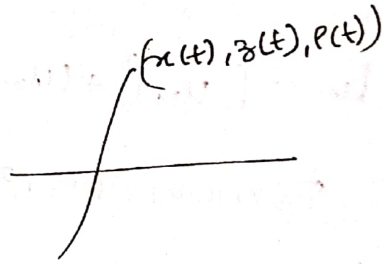
$\rightarrow \quad y = (y_1, \cdots, y_m)$

$b(y) = (b_1(y), \ldots, b_m(y))$

$$L = \sum_{i=1}^{m} b_1(y) \frac{\partial v}{\partial y_i} = 0.$$

$$g(t) = v|_\Gamma$$

$$\frac{dg}{dt} = (L(v))|_\Gamma = 0.$$



$\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \quad (x, z, p)$

$$\gamma(t) = (x_1(t), \ldots, x_n(t), z(t), p_1(t), \ldots p_n(t))$$

$$\gamma : (-t, t) \longrightarrow \mathbb{R}^{2n+1}.$$

Aim Find Γ such that $F|_\Gamma$ is a constant.

ie, $\frac{d}{dt} F(x(t), z(t), p(t)) = 0. \qquad z(t) = u(x(t))$

$$\sum_{i=1}^{n} \frac{\partial F}{\partial x_i} \frac{dx_i}{dt} + \frac{\partial F}{\partial u} \frac{dz}{dt} + \sum_{i=1}^{n} \frac{\partial F}{\partial \phi_i} \frac{d\phi_i}{dt}.$$

$$\frac{d}{dt} F(x(t), z(t), p(t)) = \sum \frac{\partial F}{\partial x_i} \frac{dx_i}{dt} + \frac{\partial F}{\partial u} \sum_{i=1}^{n} p_i(t) \frac{dx_i}{dt}$$
$$+ \sum_{i=1}^{n} \frac{\partial F}{\partial p_i} \frac{dp_i}{dt}.$$

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$$\sum_{i=1}^{n} \left[\frac{\partial F}{\partial x_i} + P_i \frac{\partial F}{\partial u} \right] \frac{dx_i}{dt} + \frac{\partial F}{\partial P_i} \frac{dP_i}{dt} = 0 .$$

$$\left.\begin{aligned} \frac{dx_i}{dt} &= \frac{\partial F}{\partial P_i} \quad 1 \le i \le n . \\ \frac{dP_i}{dt} &= -\left(\frac{\partial F}{\partial x_i} + P_i \frac{\partial F}{\partial u} \right) . \\ \frac{dz}{dt} &= \sum P_i \frac{\partial F}{\partial P_i} . \end{aligned}\right\}$$

soln to this is called bicharacteristic

$a(x) = (a_1(x), \dots, a_n(x))$, $Y = (y_1, \cdots y_n, u, P_1, \cdots, P_n) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$.

$$b(y, u, P) = (a_1(y,u,P), \dots, a_n(y,u,P), b_0(y,u,P) \dots b_n(y,u,P))$$

$$b(x, u, P) = \left(\frac{\partial F}{\partial x_1} \dots \frac{\partial F}{\partial x_n}, \sum_{i=1}^{n} P_i \frac{\partial F}{\partial P_i}, -\frac{\partial F}{\partial x_1} - P_1 \frac{\partial F}{\partial u}, \dots, -\frac{\partial F}{\partial x_n} - P_n \frac{\partial F}{\partial u} \right)$$

$a = (a_1(x), a_2(x) \dots, a_n(x))$

$b(x,u,P) = \left(a_1(x), \dots, a_n(x), \sum_{i=1}^{n} P_i a_i(x), -\frac{\partial a_1}{\partial x_1}, \dots \frac{\partial a_n}{\partial x_n} \right)$

$Lu = (u_{x_1})^2 + (u_{x_2})^2 = 1$

$F(x, u, P) = P_1^2 + P_2^2 - 1$

$b(x,u,P) = (2P_1, 2P_2, 2(P_1^2 + P_2^2), 0, 0)$.

F is constant along the bicharacteristic.

Assumption $F : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}$ smooth map

$$\begin{cases} F(0, u_0, P_0) = 0 \quad (u_0, P_0) \in \mathbb{R} \times \mathbb{R}^n . \\ \frac{\partial F}{\partial P_n} (0, u_0, P_0) \neq 0 \end{cases}$$

Given , $u_0(x', 0) = g(x')$, $u_0(0,0)\, g(0) = u_0$.

Find $P_0(x', 0)$ by the eqn .

$$F \left((x', 0), g(x), \frac{\partial g}{\partial x_1}, \dots, \frac{\partial g}{\partial x_{n-1}}, \frac{\partial F}{\partial P_n} \right) = 0 .$$

$$b(x,u,p) = \left(F_p ,\ \sum p_i \frac{\partial F}{\partial p_i} ,\ -F_x - F_u p \right).$$

F is const. along the bicharacteristic.

$$\begin{cases} \dfrac{dx}{dt} = F_p & x(0) = (x', 0) \\[2ex] \dfrac{dp}{dt} = -F_x - pF_u. & p(0) = \\[2ex] \dfrac{dz}{dt} = \sum p_i \dfrac{\partial F}{\partial p_i} \end{cases}$$

$x = x(t, x')$

$p = p(t, x')$

$z = z(t, x')$

Claim Define $u(x) = z(t, x')$. $\dfrac{\partial u}{\partial x_i} = p_i$.

Conservation Laws.

$F(x, u(x), Du(x)) = 0$

$f : \mathbb{R} \to \mathbb{R}$, smooth map.

$$\begin{cases} \dfrac{\partial u}{\partial x_2} + f'(u) \dfrac{\partial u}{\partial x_1} = 0. \\[2ex] u(x, 0) = g(x_1). \end{cases}$$

$$F\big((x_1, x_2), u, (p_1, p_2)\big) = p_2 + f'(u) p_1 = 0.$$

$$\begin{cases} \dfrac{dx_1}{dt} = f'(u) & \dfrac{dp_1}{dt} = -p_1 F_u = -p_1 f''(u). \\[2ex] \dfrac{dx_2}{dt} = 1 & \dfrac{dp_2}{dt} = -p_2 f''(u) \end{cases}$$

$$\begin{cases} \frac{\partial u}{\partial t} + f'(u)\frac{\partial u}{\partial x} = 0. & \frac{dx}{dt} = f'(u). \\ u(x,0) = g(x) \end{cases}$$

* Aim is to solve for all time $t > 0$.

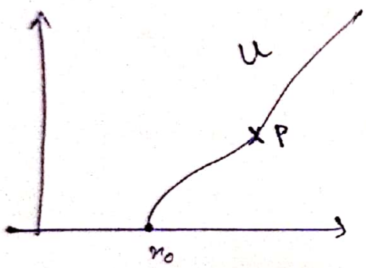
let $f'(u) = c$.

then
$$\begin{cases} \frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0. \\ u(x,0) = g(x). \end{cases}$$

$a(x_1, x_2) = (c, 1)$. $\rightarrow$ v.f.

$$\begin{cases} \frac{dx}{dt} = f'(u(x,t)) \\ x(0) = x_0. \end{cases}$$

f is smooth & derivative is bdd $\Rightarrow$ Lipschitz
$\Rightarrow$ this problem has a unique soln.



$$\frac{d}{dt}u(x(t,x_0),t) = \left(\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt}\right)(x(t,x_0),t)$$

$\frac{dx}{dt} = f' = c$, $x(0) = 0$.

Soln. $x(t, x_0) = x_0 + ct$.

$u(x_0 + ct, t) = u_0(x_0)$

Choose x_0 $\ni$ (x,t) satisfies $x = x_0 + ct \Rightarrow x_0 = x - ct$.

$u(x,t) = u_0(x_0) = u_0(x - ct)$ travelling wave soln.

Now let $f'(u) \neq c$.

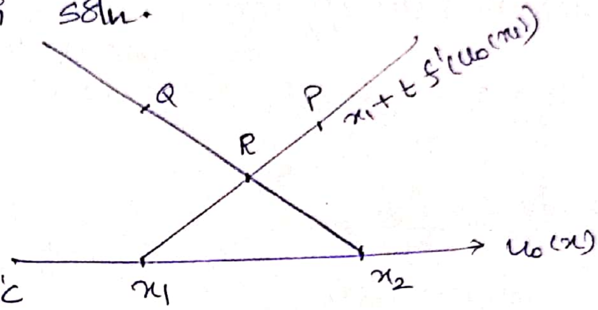
choose 2 points $x_1 < x_2$.

$\because f'(u) \neq 0$ we can choose $u_0(x) \ni$

$\alpha = f'(u_0(x_1))$ $\alpha > 0$

& $\beta = f'(u_0(x_2))$ $\beta < 0$.

Assume that $\exists$ a smooth soln.



$x_1 + t f'(u_0(x_1))$

$u(P) = u_0(x_1)$

$\because$ u is const. on characteristic

$x_2 + t f'(u_0(x_2))$

$u(Q) = u_0(x_2)$

$u(R) = u_0(x_1) = u_0(x_2)$

<u>Conclusion</u> Cannot expect a global smooth soln.

Burger's eqn when $f(u) = \frac{u^2}{2}$.

Method of ch. $\Rightarrow$ existence of soln on small nbhd.

$$\left\{\begin{array}{l} \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0 \quad \longrightarrow \quad \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) = 0. \\ \textcircled{1}-\left\{\begin{array}{l} \frac{d}{dx}\left(\frac{u^2}{2}\right) = 0 \quad \to \quad u \frac{du}{dx} = 0 \quad \text{if } u \text{ is smooth} \\ u(0) = 1 \quad \Rightarrow u(x) \equiv 1 \quad \forall x. \end{array}\right. \end{array}\right.$$

$\longrightarrow$ conservative form

$$u(x) = \begin{cases} 1 & x < 1 \\ -1 & x \geq 1 \end{cases} \quad \to \text{ is a soln to } \textcircled{1}$$

but u is not continuous.

Similarly we get inifinitely many solns to ① which are discontinuous.

$$\begin{cases} u\dfrac{du}{dx} = 0. \\ u(0) = 1 \end{cases}$$ Non conservative form.

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial}{\partial x} f(u) = 0 \\ u(x,0) = u_0(x) \quad x \in \mathbb{R}. \end{cases}$$ → Conservative.

$$\begin{cases} \dfrac{\partial u}{\partial t} + f'(u)\dfrac{\partial u}{\partial x} = 0 \\ u(x,0) = u_0(x) \end{cases}$$ non conservative.

- $$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \dfrac{\partial^2 u}{\partial x^2} = 0 \\ u(x,0) = g(x) \\ u_t(x,0) = 0. \end{cases}$$ $u(x,t) = \frac{1}{2}\left(g(x+ct) + g(x-t)\right)$

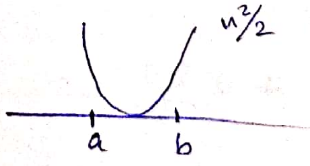
u_0 is not diff. eventhough D'Alembert's soln is actually a soln. (Distribution thry gives meaning)

Riemann problem:

$$\frac{\partial u}{\partial t} + f'(u)\frac{\partial u}{\partial x} = 0.$$

$$u(x,0) = u_0(x)$$

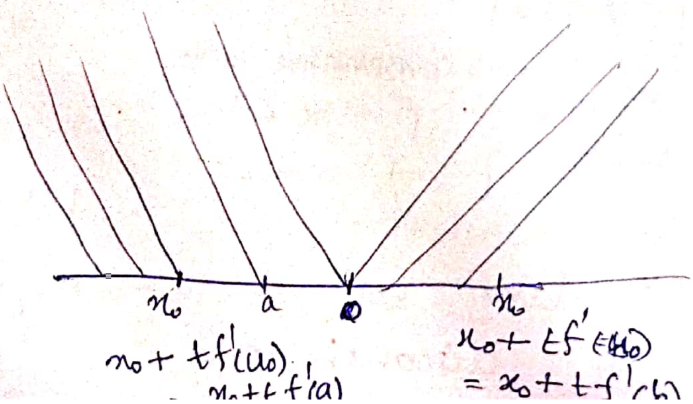
$$u_0(x) = \begin{cases} a & x < 0 \\ b & x > 0. \end{cases}$$



let $f(u) = \dfrac{u^2}{2}$

Case 1 $a < b$.

$f'(a) < 0 \quad f'(b) > 0.$



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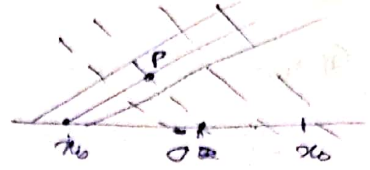
Case 2 $a > b$. $f'(a) > 0$ $f'(b) < 0$.



$u(P) = a = b$.

but $a \neq b$ →←

- How to overcome this difficulty ??

☺ Distributions.

Let $\phi \in C_0^\infty(\mathbb{R} \times (0,\infty))$

$$0 = \int_{-\infty}^{\infty} \int_0^{\infty} \left(\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(x) \right) \phi(x,t)\, dt\, dx.$$

consider $\int_0^\infty \frac{\partial u}{\partial t} \phi(x,t)\, dt$, $\int_0^\infty \frac{\partial u}{\partial t} \phi(x,t)\, dt = \lim_{m\to\infty} -\int_0^m u \phi_t\, dt + u(x,t)\phi(x,t)\Big|_0^m$

$$= -\int_0^\infty u \phi_t\, dt - u(x,0)\phi(x,0).$$

$$0 = \int_{-\infty}^{\infty} \int_0^{\infty} \left(u \frac{\partial \phi}{\partial t} + f(u) \frac{\partial \phi}{\partial x} \right) dx\, dt + \int_{-\infty}^{\infty} u_0(x) \phi(x,0)\, dx.$$

$\forall \phi \in C_0^\infty(\mathbb{R} \times [0,\infty))$

——— $(*)$

Weak soln.

$u \in L^\infty(\mathbb{R} \times [0,\infty))$ is called a weak soln if $(*)$ holds

$\forall \phi \in C_0^\infty(\mathbb{R} \times [0,\infty))$

? Existence of weak soln.

? multiplicity of weak soln.

? concept of entropy.

? uniqueness of ~~weak~~ entropy soln.

? Derivation of Numerical scheme.

PSD

Green's Identity

If $u, v \in C^2(\bar{\Omega})$ then

$$\int_{\Omega} (u \Delta v - v \Delta u) = \int_{\partial\Omega} \left(u \frac{\partial v}{\partial \nu} - v \frac{\partial u}{\partial \nu}\right)$$

$$u \Delta v = \operatorname{div}(u \nabla v) - \nabla u \cdot \nabla v .$$

$$\int_{\Omega} u \Delta v = \int_{\partial\Omega} u \frac{\partial v}{\partial \nu} - \int_{\Omega} \nabla u \cdot \nabla v \quad \text{by div.thm.}$$

$||^{ly}$,

$$\int_{\Omega} v \Delta u = \int_{\partial\Omega} v \frac{\partial u}{\partial \nu} - \int_{\Omega} \nabla u \cdot \nabla v .$$

Laplace operator Δ is invariant under translations and rotations.

$$x \longmapsto \underbrace{x+a}_{y} .$$

$$\Delta_x = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2} = \Delta_y .$$

$y = Ax$ $\quad$ A is an orthogonal matrix (rotation)

Check $\Delta_x = \Delta_y$.

Look for a soln u of $\Delta u = 0$ of the form $u(x) = v(r)$

$r = |x|$.

$\Delta u = 0$ becomes $\quad v'' + \frac{n-1}{r} v' = 0 \quad r > 0$,

$$\Rightarrow \left(r^{n-1} v'\right)' = 0 .$$

$$\Rightarrow r^{n-1} v' = \text{Const} .$$

$$v(r) = \begin{cases} \dfrac{\text{Const}}{r^{n-2}} & n > 2 \\ \text{Const} . \log r & n = 2 \end{cases}$$

let $\Phi(x) = \begin{cases} \dfrac{C_n}{|x|^{n-2}} & n>2 \\ C_2 \log|x| & n=2 \end{cases} \quad x \neq 0.$

where $C_n = -\dfrac{1}{(n-2)\,\sigma_n}$ $(n>2)$, $C_2 = -\dfrac{1}{2\pi}$

Φ is called the fundamental soln of Δ.

- $\Phi \in C^\infty(\mathbb{R}^n \setminus \{0\})$ and Φ is harmonic in that region
- Φ is symmetric ie, $\Phi(x) = \Phi(-x)$
- $\Phi, \dfrac{\partial \Phi}{\partial x_i}$ are locally integrable

But $\dfrac{\partial^2 \Phi}{\partial x_i \partial x_j}$ is not locally integrable.

$\displaystyle\int_{B_1(0)} |\Phi(x)|\,dx < \infty$

$x = r\omega \quad |\omega| = 1.$

$\omega = (\omega_1, \omega_2)$

$\omega = \cos^2\theta + i \sin^2\theta$

$dx = r^{n-1}\,dr\,ds(\omega)$

$\displaystyle\int_{B_1(0)} \frac{dx}{|x|^{n-2}} = \int_0^1 \frac{r^{n-1}}{r^{n-2}}\,dr$

$r^2 = |x|^2 = x_1^2 + x_2^2 + \dots + x_n^2$

$|x_i| \leq |x|.$

$\displaystyle\int_{B_1(0)} \left|\frac{\partial \Phi}{\partial x_i}\right| dx = \int_{B_1(0)} \text{Const.} \frac{|x_i|}{|x|^n}\,dx \leq \frac{\text{Const}}{|x|^{n-1}}$

$\dfrac{\partial^2 \Phi}{\partial x_i \partial x_j} = \text{Const.} \left|\dfrac{x_i x_j}{|x|^{n+2}}\right| \leq \dfrac{\text{Const}}{|x|^n}$

For any fixed $y \in \mathbb{R}^n$ $\Phi(x-y)$ is harmonic in $\mathbb{R}^n \setminus \{y\}$

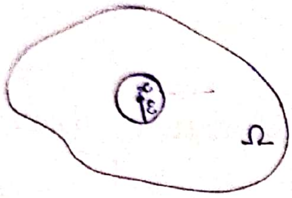
<u>lemma</u> ~~let $u \in C^2(\bar{\Omega})$. Therefore~~

Lemma

Let $u \in C^2(\bar{\Omega})$. Then, for any $x \in \Omega$ we've

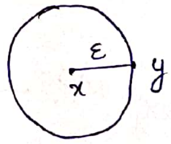
$$u(x) = \int_{\Omega} \Phi(x-y)\,\Delta u(y)\,dy + \int_{\partial\Omega} u(y)\frac{\partial \Phi}{\partial \nu}(y-x)\,ds(y)$$

$$-\int_{\partial\Omega} \Phi(y-x)\frac{\partial u}{\partial \nu}(y)\,ds(y).$$



Apply Green's identity to $\Phi(x-y)$ and $u(y)$ in $\Omega \setminus B_\varepsilon(x)$.

$$\int_{\Omega\setminus B_\varepsilon(x)} \left(u\,\Delta\Phi - \Phi(x-y)\,\Delta u(y)\right) dy.$$



$$\nu = \frac{x-y}{|x-y|} = \frac{x-y}{\varepsilon}.$$

$$\frac{\partial \Phi}{\partial \nu} = \nabla\Phi \cdot \nu$$

$$= \frac{1}{\sigma_n}\,\frac{x_i - y_i}{|x-y|^n}\,\frac{(x_i-y_i)}{\varepsilon}$$

$$= \frac{1}{\sigma_n}\,\frac{|x-y|^2}{\varepsilon^{n+1}} = \frac{1}{\sigma_n\,\varepsilon^{n-1}}$$

$$\int_{|y-x|=\varepsilon} u(y)\frac{\partial\Phi}{\partial\nu}(y-x)\,ds = \frac{1}{\sigma_n\,\varepsilon^{n-1}}\int_{|y-x|=\varepsilon} u(y)\,ds(y)$$

$$\longrightarrow u(x) \text{ as } \varepsilon \to 0.$$

$$\left|\int_{|y-x|=\varepsilon} \Phi(y-x)\frac{\partial u}{\partial \nu}(y)\,ds(y)\right| \leq \frac{\text{Const.}\,\varepsilon^{n-1}}{\varepsilon^{n-2}} \to 0 \text{ as } \varepsilon \to 0.$$

$|\Phi(y-x)|$ on $|y-x| = \varepsilon$ $= \dfrac{C_n}{\varepsilon^{n-2}}$.

$\dfrac{\partial u}{\partial \nu}$ on $|y-x| = \varepsilon$ is bdd, $\because u \in C^2(\bar{\Omega})$

$$\int_{|y-x|=\varepsilon} ds(y) = \varepsilon^{n-1}$$

$$\int_{-\varepsilon}^{\varepsilon} dy = y\Big|_{-\varepsilon}^{\varepsilon} = 2\varepsilon$$

Suppose $u \in C_c^2(\Omega)$

then. $u(x) = \int_\Omega \Phi(x-y)\, \Delta u(y)\, dy$.

$\Rightarrow \quad \Delta \Phi = \delta_0$.

(Suppose $u \in C_c^\infty(\mathbb{R}^n)$), Φ is locally integrable

$$\Phi(\psi) = \int \Phi(x)\, \psi(x)\, dx$$

$$\frac{\partial \phi}{\partial x_i}(\psi) = -\int \Phi(x) \frac{\partial \psi}{\partial x_i}\, dx.$$

$$\Delta \Phi(\psi) = \int \Phi(x)\, \Delta \psi\, dx = \psi(0) = \delta_0(\psi).$$

Lemma Let $u \in C_c^2(\mathbb{R}^n)$. Then

$$u(x) = \int_{\mathbb{R}^n} \Phi(x-y)\, \Delta u(y)\, dy = \Phi * \Delta u.$$

MVP $\quad u(x) = \frac{1}{\sigma_n R^{n-1}} \int_{\partial B_R(x)} u(y)\, ds(y).$

$\because \int_\Omega \Delta u\, dx = \int_{\partial\Omega} \frac{\partial u}{\partial \nu}\, ds$ & $\Delta u = 0$.

Lemma $\Rightarrow \quad u(x) = \int_{\partial\Omega} u(y) \frac{\partial \Phi}{\partial \nu}(y-x)\, ds(y)$

Choose $\Omega = B_R(x)$

$$= \frac{1}{\sigma_n R^{n-1}} \int_{\partial B_R(x)} u(y)\, ds(y).$$

Weak maximum principle.

Let $u \in C(\bar{\Omega}) \cap C^2(\Omega)$. and $\Delta u \geq 0$ in Ω.

Then $\max_{\bar\Omega} u = \max_{\partial\Omega} u.$

2) If $\Delta u \leq 0$ then $\max_{\bar{\Omega}} u = \min_{\partial\Omega} u$.

3) If $\Delta u = 0$, then $\max_{\bar{\Omega}} |u| = \max_{\partial\Omega} |u|$.

Uniqueness.

There is atmost one soln of the problem

$$\Delta u = f \quad \text{in } \Omega$$
$$u = g \quad \text{on } \partial\Omega.$$

If u_1, u_2 are 2 solns then $u = u_1 - u_2$ satisfies

$\Delta u = 0$ in Ω & $u = 0$ on $\partial\Omega$.

weak max principle (3) $\Rightarrow$ $\max_{\bar{\Omega}} |u| = \max_{\partial\Omega} |u| = 0$ $\because u = 0$ on $\partial\Omega$

$\Rightarrow |u| = 0 \Rightarrow u = 0 \Rightarrow u_1 = u_2$.

Proof

First suppose $\Delta u > 0$ in Ω. If $x_0 \in \Omega$ is a point of maximum of u, then $\Delta u(x_0) \leq 0$.

• Now suppose $\Delta u \geq 0$.

Let $v(x) = u(x) + \varepsilon |x|^2$.

$$\Delta v = \Delta u + 2\varepsilon \geq 2\varepsilon > 0.$$

$\Rightarrow \max_{\Omega} v = \max_{\partial\Omega} v$.

$\because u \leq v$ $\max_{\bar{\Omega}} u \leq \max_{\bar{\Omega}} v$.

On the other hand

$$v(x) = u(x) + \varepsilon |x|^2 \leq \max_{\partial\Omega} u + c\varepsilon.$$

2 dim. $u_{x_1x_1} + u_{x_2x_2} = 0$ in $x_2 > 0$.

$u = 0$ on $x_2 = 0$.

$u(x_1, x_2) = x_2$.

Strong maximum principle.

Ω is connected and u is harmonic in Ω

then if $M = \max_{\bar{\Omega}} u$, $u(x) < M \quad \forall x \in \Omega$

or $u \equiv M$.

Proof

Suppose $x_0 \in \Omega \ni u(x_0) = M$.

To show $u \equiv M$,

let $\tilde{\Omega} = \{ y \in \Omega : u(y) = M \}$

Then $\tilde{\Omega}$ is closed by continuity of u.

Let $y \in \tilde{\Omega}$. choose $r > 0 \ni \overline{B_r(y)} \subset \Omega$.

By MVP,

$$u(y) = \frac{n}{\sigma_n R^n} \int_{B_R(y)} u(x)\,dx.$$

$(u(x)) \to M$.

$$M = u(y) = \frac{n}{\sigma_n R^n} \int_{B_R(y)} u(x)\,dx.$$

$$\frac{n}{\sigma_n R^n} \int_{B_R(y)} (u(y) - u(x)) = 0.$$

$u(x) = u(y) = M \quad \forall x \in B_R(y)$.

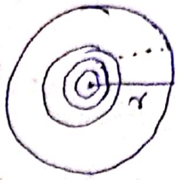
$\Rightarrow B_R(y) \subset \tilde{\Omega}$.

... and closed. $\Rightarrow \tilde{\Omega} = \Omega$

MVT proof

Suppose $u(x) = \frac{1}{\sigma_n r^{n-1}} \int_{\partial B_r(x)} u(y)\,ds(y) \quad \forall r > 0.$

$$\int_{B_r(x)} u(y)\,dy = \int_0^r dt \int_{\partial B_t(x)} u(y)\,ds(y).$$



$$= \sigma_n \int_0^r t^{n-1}\,dt \underbrace{\frac{1}{\sigma_n t^{n-1}} \int_{\partial B_t(x)} u(y)\,ds(y)}_{u(x)}$$

$$= \frac{\sigma_n r^n}{n} u(x).$$

Neumann problem

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = g & \text{on } \partial\Omega \end{cases}$$

If u_1, u_2 are any 2 solns they differ by a const.

Let $u = u_1 - u_2$.

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases}$$

We've $\int_\Omega u \Delta u = -\int_\Omega |\nabla u|^2 dx + \int_{\partial\Omega} u \frac{\partial u}{\partial \nu}$

$\Rightarrow \int_\Omega |\nabla u|^2 = 0.$

$\nabla u = 0.$

$u = c.$

$u_1 = c + u_2$

Suppose u is a soln

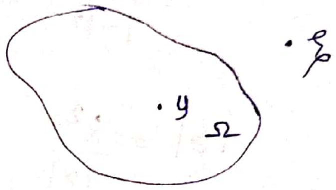
$$\int_\Omega f = \int_\Omega \Delta u = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} = \int_{\partial\Omega} g . \qquad \text{div. thm.}$$

$$* \begin{cases} \Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

$$u(x) = \int_\Omega \Phi(x-y)\, \Delta u(y)\, dy + \int_{\partial\Omega} u(y) \frac{\partial \Phi}{\partial \nu}(y-x)\, ds(y)$$

$$- \int_{\partial\Omega} \Phi(y-x) \frac{\partial u}{\partial \nu}(y)\, ds(y). \quad \text{—} (1).$$

Fix $\xi \notin \bar{\Omega}$



Consider $\Phi(\xi - y)$, $y \in \Omega$.

Then Φ is a harmonic fn.

Then Green's identity applied to $u(y)$ & $\Phi(\xi - y)$.

$$0 = \int_\Omega \Phi(\xi - y)\, \Delta u(y)\, dy + \int_{\partial\Omega} u(y) \frac{\partial \Phi}{\partial \nu}(\xi - y)$$

$$- \int_{\partial\Omega} \Phi(\xi - y) \frac{\partial u}{\partial \nu} . \quad \text{—} (2).$$

Let $d : (\bar{\Omega})^c \longrightarrow \mathbb{R}$ be any const. fn Multiplying (2) by $d(\xi)$ and subtract from (1)

$$u(x) = \int_\Omega \left(\Phi(x-y) - d(\xi) \Phi(\xi - y) \right) \Delta u(y)\, dy$$

$$+ \int_{\partial\Omega} \frac{\partial u}{\partial \nu}(y) \left[d(\xi) \Phi(\xi - y) - \Phi(x-y) \right] ds(y)$$

$$+ \int_{\partial\Omega} u(y) \frac{\partial}{\partial \nu} \left(\Phi(x-y) - d(\xi) \Phi(\xi - y) \right) ds.$$

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If we can choose d $\ni$

$$d(\xi)\,\Phi(\xi-y)-\Phi(x-y)=0. \qquad (*)$$

$\forall\, y\in\partial\Omega$, then (3) simplifies into a nice formula.

For any $x\in\Omega$, we should be able to find a $\xi=\xi(x)\notin\bar{\Omega}$ $\ni$ $(*)$ holds.

Let $\Omega=B_R(0)$

& $\xi=\dfrac{R^2}{|x|^2}x$ Kelvin transform.

$$|\xi|=\frac{R^2}{|x|}>R.$$

If $|y|=R$ then $|y-\xi|=\dfrac{R}{|x|}|x-y|.$

Fundamental soln involves only distances.

$$|y|=R \qquad \frac{\partial}{\partial\nu}\left(\frac{1}{|x-y|^{n-2}}\right)=\nabla_y\left(\frac{1}{|x-y|^{n-2}}\right)\cdot\nu(y).$$

$$=\frac{n-2}{R|x-y|^n}(x-y,y) \text{ dot product}.$$

$$=\frac{(n-2)((x,y)-R^2)}{R|x-y|^n}.$$

<u>Theorem</u>

Let $\Omega=B_R(0)$ & $u\in C^2(\bar{\Omega})$. Then for any $x\in\Omega$, we've

$$u(x)=\int_{|y|=R}P_R(x,y)\,u(y)\,ds(y)+\int_{|y|<R}G_R(x,y)\,\Delta u(y)\,dy.$$

$$P_R(x,y) = \frac{R^2 - |x|^2}{\sigma_n R |x-y|^n} \rightarrow \text{Poisson Kernel.}$$

$$G_R(x,y) = \frac{1}{\sigma_n}\left[\frac{1}{|x-y|^{n-2}} - \frac{(R/|x|)^{n-2}}{|(R/|x|)^2 x - y|^{n-2}}\right]$$

$\rightarrow$ Green's f^n.

Theorem

let $\Omega = B_R(0)$ and $g \in C(\partial\Omega)$. Define

$$u(x) = \int_{|y|=R} P_R(x,y)\, g(y)\, ds(y). \quad x \in \Omega.$$

(P.f) —

Then u is harmonic and $u = g$ on $\partial\Omega$.

$$\lim_{x \to y \in \partial\Omega} u(x) = g(y).$$

Poisson's formula — (P.f) $\begin{cases} \Delta u = 0 & \text{in } \Omega = B_R(0) \\ u = g & \text{on } \partial\Omega \end{cases}$

$R^2 - |x|$

$$\int_{\partial B_R(0)} P_R(x,y)\, ds(y) = 1 \Leftarrow \text{let } u(x) \equiv 1 \text{ in the above thm.}$$

$\Delta u = f$ in Ω

$u = g$ on $\partial\Omega$. can be splitted into 2 problems

$\begin{cases} \Delta u_1 = f & \text{in } \Omega \\ u_1 = 0 & \text{on } \partial\Omega \end{cases}$ & $\begin{cases} \Delta u_2 = 0 & \text{in } \Omega \\ u_2 = g & \text{on } \partial\Omega. \end{cases}$

Then $\exists$ continuous f^ns f $\ni$ no C^2 soln u exists satisfying $\Delta u = f$.

• Suppose $f \in C^1(\bar{\Omega})$. Define $u(x) = \int_\Omega \Phi(x-y) f(y)\, dy$. $x \in \Omega$

f is a Holder continuous f^n $\longrightarrow$ Schauder theory.

Let u be a harmonic f^n in $\mathbb{R}^n$ and assume u is semi-bdd ($u \geq M$ or $u \leq M$)

Assume $u \geq 0$.

Let $x \in \mathbb{R}^n$ and choose $R > |x|$.

Apply p.f in $B_R(0)$.

$$u(x) = \int_{|y|=R} P_R(x,y)\, u(y)\, ds(y)$$

$$|x-y| \leq |x| + |y| = R + |x|.$$
$$|x-y| \geq |y| - |x| = R - |x|$$

$$\frac{R^2 - |x|^2}{\sigma_n R (R+|x|)^n} \leq P_R(x,y) \leq \frac{R^2 - |x|^2}{\sigma_n R (R-|x|)^n}$$

$$\frac{R^2 - |x|^2}{\sigma_n R (R+|x|)^n} \int_{|y|=R} u(y)\, ds(y) \leq u(x) \leq \frac{R^2 - |x|^2}{\sigma_n R (R-|x|)^n} \int_{|y|=R} u(y)\, ds(y)$$

$$\frac{R^{n-1}(R^2 - |x|^2)}{\sigma_n R (R+|x|)^n} u(0) \leq u(x) \leq \frac{R^{n-1}(R^2-|x|^2)}{R(R-|x|)^n} u(0).$$

Take $R \to \infty$ then $u(x) = u(0)$

Liouville's theorem $\Rightarrow u \equiv \text{Const}.$

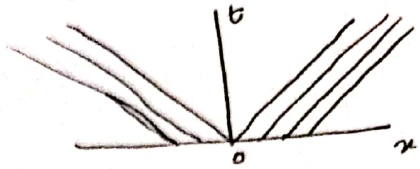
22/12/19

AA

Global existence.

(1) $\begin{cases} \dfrac{\partial u}{\partial t} + f'(x)\dfrac{\partial u}{\partial x} = 0. & x \in \mathbb{R}, t>0 \\ u(x,0) = u_0(x) \end{cases}$

Riemann problem



$a<b$

$f'(a)<0 \quad f'(b)>0.$

find the characteristics.

→ Intersecting characteristics.

→ No global soln for C^1 function.

→ Weak formulation.

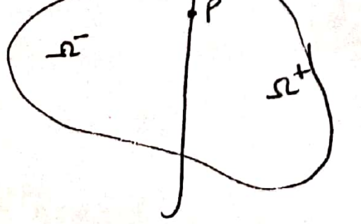
$$0 = \int_{-\infty}^{\infty}\int_{0}^{\infty}\left[u\frac{\partial \phi}{\partial t} + f(x)\frac{\partial \phi}{\partial x}\right]dt\,dx + \int_{-\infty}^{\infty} u_0(x)\phi(x,0)\,dx.$$

* Jump Condition or Rankin – Hugoniot Condition.

Theorem. u is a weak soln. in Ω iff u satisfies the following (R–H) condition $\forall P \in \Gamma$. $\Gamma = \{(x(t),t)\}$

$$(u^+(P) - u^-(P))\frac{dx}{dt}(P) = f(u^+(P)) - f(u^-(P))$$

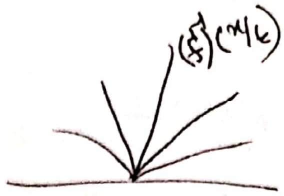
shock curve : A curve of discontinuity.



To prove apply Green's theorem on Ω^- & Ω^+. Add the results.

We want to reduce (1) to a one variable case inorder to solve it. For that we look for the symmetries of the eqn (1).

Let $\alpha>0$ Put $X = \alpha x$ for $\alpha>0$. $T = \alpha t$.

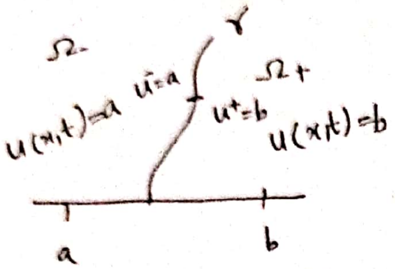


$$\frac{X}{T} = \frac{x}{t} = \xi.$$

$$u(x,t) = \Psi(\xi) = \Psi(x/t)$$

$$\Psi(x/t) = (f')^{-1}(x/t)$$ → Rarefaction wave

→ this is a soln to (1)



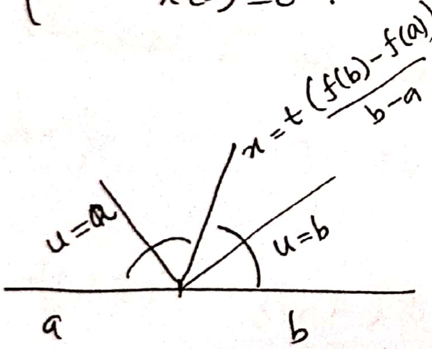
$a>b$.

by R-H, $(b-a)\frac{dx}{dt} = f(b)-f(a)$

$x(0)=0$

$$\begin{cases} \dfrac{dx}{dt} = \dfrac{f(b)-f(a)}{b-a} \\ x(0)=0. \end{cases}$$

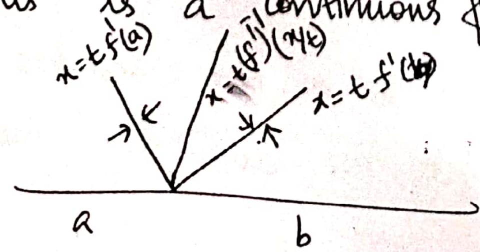
Soln $x = t\left(\dfrac{f(b)-f(a)}{b-a}\right)$



We've chosen an arbitrary γ ~~but~~ the soln is a straightline.

$$u(x,t) = \begin{cases} a, & x<tf'(a) \\ (f')^{-1}(x/t), & tf'(a) < x < tf'(b) \\ b, & x > tf'(b). \end{cases}$$

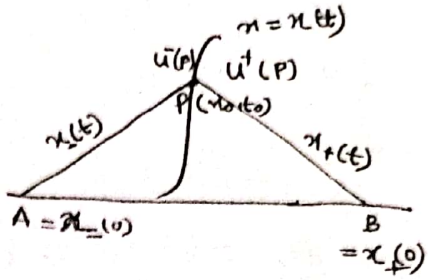
this is a continuous fn.



apply R-H conditions by taking $\gamma_1 = (x = tf'(a))$ & $\gamma_2 = (x = tf'(b))$

Assume, $a<b$ $\quad f'(a) < \frac{f(b)-f(a)}{b-a} < f'(b)$

Entropy Hypothesis. • u be a weak soln.



$$x_-(t) = x_0 + (t-t_0) f'(u^-(P))$$

$$x_+(t) = x_0 + (t-t_0) f'(u^+(P))$$

$P(x_0,t_0)$ point of discontinuity.

$$A = x_-(0) = x_0 - t_0 f'(u^-(P))$$

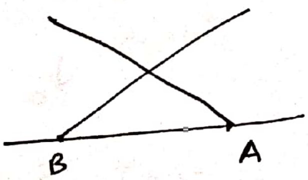
$$B = x_+(0) = x_0 - t_0 f'(u^+(P))$$

$\boxed{u \text{ is an entropy soln if } A < B.}$

$\Rightarrow f'(u^+(P)) < f'(u^-(P)) \Rightarrow u^+(P) < u^-(P)$.

(Applying method of ch. we get several solns for the problem. Inorder to choose a soln from that we need one more condit^n. For that we study the points of discontinuity)

If $A > B$



• f is convex
• Riemann problem.

CFL condition.

PSD

• Poisson formula when $\Omega = B_R(0)$
• $\Omega = \mathbb{R}^n_+$

General domains
→ Perron's method → Fritz John.

$\Delta u = f$ in Ω

$u = 0$ on $\partial\Omega$.

Representation of sol^n via Newtonian potential.

- $f \in C_c^2(\mathbb{R}^n)$
- $f \in C^1(\bar{\Omega})$

Poisson's formula for $\mathbb{R}^n_+$.

Poisson's kernel;

$$K(x,y) = \frac{2x_n}{\sigma_n |x-y|^n}$$

$x \in \mathbb{R}^n_+$, $y \in \partial\mathbb{R}^n_+$.

If g is a bdd continuous f^n in $\mathbb{R}^{n-1}$.

Define, $u(x) = \int_{\partial\mathbb{R}^n_+} K(x,y) g(y) dy$ $x \in \mathbb{R}^n_+$.

Then u is harmonic in $\mathbb{R}^n_+$ and

$$\lim_{x \to y \in \partial\mathbb{R}^n_+} u(x) = g(y).$$

* MVP $\Rightarrow$ harmonicity

■ Let $f \in C_c^2(\mathbb{R}^n)$. Define

$$u(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y) dy$$

$$= \Phi * f = f * \Phi$$

(This cannot be done in bdd domains)

Then u is bdd & satisfies $\Delta u = f$. $n \geq 3$.

* $D^\alpha(f*g) = D^\alpha f * g$

If $f \in L^p$, $g \in L^q$ $\frac{1}{p} + \frac{1}{q} \geq 1$

then $f*g \in L^r$

where $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$

For $n \geq 3$ $\Phi(x) \to 0$ as $|x| \to \infty$.

■ Proof

$$u(x) = \int_{\mathbb{R}^n} \Phi(y) f(x-y) dy$$

$$\Delta u(x) = \int_{\mathbb{R}^n} \Phi(y) (\Delta f)(x-y) dy \equiv I.$$

$$= \Phi * \Delta f.$$

$$I = \int_{B_\varepsilon(0)} \Phi(y) \underbrace{\Delta f(x-y)}_{\text{bdd}} dy + \int_{\mathbb{R}^n \setminus B_\varepsilon(0)} \Phi(y) \Delta f(x-y)$$

$\downarrow$ 0 as $\varepsilon \to 0$ $\qquad$ $\downarrow$ integration by parts.

$$= -\int_{\mathbb{R}^n \setminus B_\varepsilon(0)} \nabla\Phi(y) \nabla f(x-y) dy + \int_{\partial B_\varepsilon(0)} \Phi(y) \frac{\partial f}{\partial \nu}(x-y) ds_y$$

$\downarrow$ integration by parts $\qquad$ $\downarrow$ 0 as $\varepsilon \to 0$.

$$= \int_{\mathbb{R}^n \setminus B_\varepsilon(0)} \Delta\Phi(y) f(x-y) dy - \int_{\partial B_\varepsilon(0)} \frac{\partial \Phi}{\partial \nu}(y) f(x-y)$$

$\downarrow$ 0 $\because \Phi$ is harmonic $\qquad$ $\downarrow$ $f(x)$ as $\varepsilon \to 0$.

$= f(x)$ as $\varepsilon \to 0$

Theorem

If $f \in C_c^2(\mathbb{R}^n)$ then any bdd sol^n of

$$\Delta u = f \text{ in } \mathbb{R}^n$$

is given by

$$u(x) = \underbrace{\int_{\mathbb{R}^n} \Phi(y) f(x-y) dy}_{\tilde{u}} + \text{Const}$$

Proof.

If u is any bdd sol^n of $\Delta u = f$, then $u - \tilde{u}$ is bdd harmonic f^n. Hence a constant

Let $f \in C^1(\bar\Omega)$.

Define $u(x) = \int_\Omega \Phi(x-y) f(y)\,dy \quad x \in \Omega$ —($\blacktriangle$)

Then $\Delta u = f$ in Ω.

Integrals of potential type

Let $f \in L^\infty(\Omega)$

Define $u(x) = \int_\Omega \frac{f(y)}{|x-y|^\alpha}\,dy \quad x \in \Omega$

$\alpha > 0$.

This integral is well defined if $\alpha < n$.

- u is continuous.
- If $\alpha + 1 < n$, then u is C^1.
- If $\alpha + k < n$, then u is C^k.

Proof. $u \in C^1$ and

$$\frac{\partial u}{\partial x_i} = \int_\Omega \frac{\partial \Phi}{\partial x_i}(x-y) f(y)\,dy. \qquad \frac{\partial \Phi}{\partial x_i} = -\frac{\partial \Phi}{\partial y_i}$$

$$= \int_\Omega \Phi(x-y) \frac{\partial f(y)}{\partial y_i}\,dy \to C^1$$

$$- \int_{\partial\Omega} \Phi(x-y) f(y) \nu_i(y)\,ds(y)$$

$\downarrow C^\infty$ $\quad$ $\downarrow$ i^{th} coordinate of ν.

$\Rightarrow \frac{\partial u}{\partial x_i}$ is C^1.

$\Rightarrow u \in C^2(\Omega)$.

- Let $\psi \in C_c^2(\Omega)$ $\quad$ $u(x) = \int_{\mathbb{R}}$ choose $f = \Delta u$ in ($\blacktriangle$)

$\Rightarrow \Delta(\Phi * \psi) = \psi$. $\quad \Rightarrow \Phi * \Delta u = u$.

i.e, $\Phi * \Delta\psi = \psi$

Consider $\int_\Omega (\Delta u)\psi = \int_\Omega u \Delta\psi$.

no bdry term $\because \psi$ has compact support.

$$\int_\Omega (\Delta u)\psi = \int_\Omega u \Delta\psi$$

$$= \int_\Omega \Delta\psi \left(\int_\Omega \Phi(x-y) f(y)\,dy \right) dx$$

$$= \int_\Omega f(y) \int_\Omega \Phi(x-y) \Delta\psi\,dx\,dy$$

$$= \int_\Omega f(y)\psi(y)\,dy \quad \forall \psi$$

$$\Rightarrow \int_\Omega (\Delta u - f)\psi = 0$$

$\because C_c^2(\Omega)$ is dense in $L^2(\Omega)$.

$\Delta u - f = 0$ a.e. in f.

$\Rightarrow \Delta u = f$ in Ω.

Unbdd domains

Lemma

Let u be a bdd harmonic f^n in $\mathbb{R}^2_+$. Then

$$\sup_{\mathbb{R}^2_+} |u| = \sup_{\partial\mathbb{R}^2_+} |u|.$$

Lemma Let $u \in C^2(\mathbb{R}^n)$ satisfying

$\Delta u \geq \lambda u$, where $\lambda > 0$.

If u is bdd then $u \equiv 0$.

Proof. First prove that if u bdd above then $u \leq 0$.

Suppose not. Let $u > 0$ somewhere.

Suppose u attains a +ve max at x_0.

Then $u(x_0) > 0$ and $\Delta u(x_0) \leq 0$. $\rightarrow\leftarrow$

Otherwise

consider the f^n $v(x) = \frac{u(x)}{\xi(x)}$

$\xi(x) = \cosh \varepsilon|x|$, $\varepsilon > 0$.

$\xi > 0$.
$\xi \in C^{\infty}(\mathbb{R}^n)$.

$\therefore v$ attains a +ve max.

$\Delta u = \Delta(\xi v) = \xi \Delta v + v \Delta \xi + 2 \nabla \xi \nabla v$.

$\Delta \xi = \varepsilon^2 \xi + \varepsilon(n-1) \frac{\sinh \varepsilon|x|}{|x|}$

$\lambda u \le \Delta u$ becomes

$\lambda v \le \Delta v + 2\left(\frac{\nabla \xi}{\xi} \cdot \nabla v\right) + v\Big(\varepsilon^2 + \varepsilon(n-1) \underbrace{\frac{\tanh \varepsilon|x|}{|x|}}_{\text{bdd f}^n}$.

let x_0 be a point of +ve max for v.

AA

Given $u_0 \in C_c^{\infty}(\mathbb{R})$

Let $h > 0$.

Godunov's scheme.
→ 3 point scheme.

Weierstrass approximat^n thm Proof.

Viscosity method.

$$\begin{cases} \frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) = \varepsilon u_{xx}. \\ u(x,0) = u_0(x). \end{cases}$$

$\varepsilon \to 0$?

Kruzkov. (1960)
$u^{\varepsilon} \to u \in L^1_{loc}$
u is a smooth f^n.

RS

Motivation.

Weak soln for PDEs.

1) Burger's eqn.

$$\begin{cases} u_t + u u_x = 0 \\ u(x,0) = u_0(x). \end{cases}$$

Arises in gas dynamics, fluid dynamics, traffic flow problem.

Viscous Burger's eqn

$u_t + \gamma u_{xx} + u u_x = 0$.
γ ↓ Viscosity coefficient.

Soln exists for short time. by method of characteristics

- Need for generalisation of solns.

2) $\Omega \subseteq \mathbb{R}^n$ with smooth bdry $\partial\Omega$

Poisson eqn,

$$-\Delta u = f \text{ in } \Omega$$
$$u = 0 \text{ on } \partial\Omega.$$

(Potential energy) internal energy, Applied energy

Let $J(v) = \frac{1}{2}\int_{\Omega} |\nabla v|^2 dx - \int_{\Omega} f v \, dx$

Find $u \in \mathcal{A}$ (some class of f^n) which minimizes $J(v)$.
→ energy f^{n}al.

$\nabla v = \left(\frac{\partial v}{\partial x_1}, \ldots, \frac{\partial v}{\partial x_n}\right)$

physical - energy.

3) $\begin{cases} a(x)\dfrac{d^2u}{dx^2} = 1 \quad \text{in } (0,1) \\ u(0) = u(0) = u(1) = 0. \end{cases}$

If $a(x) = \begin{cases} 1 & 0 \le x \le 1/2 \\ 2 & 1/2 < x < 1. \end{cases}$

then soln doesn't exist.

→ need for generalisation of # solns.

4) $\frac{d}{dx}(a(x)\,u'(x)) = 1$

$u(0) = u(1) = 0.$

We don't get classical soln. with $a(x)$ defined above.

(*) $\begin{cases} \text{Consider } u''(x) + u(x) = f \\ \text{in } (a,b) \text{ with } u(a) = u(b) = 0. \end{cases}$

<u>classical soln</u>

A fn $u \in C^2[a,b]$ satisfying the above problem.

let $f \in C[a,b]$

let $\phi \in C_0^2(a,b)$

$$\int_a^b u''(x)\phi(x)\,dx + \int_a^b u\phi\,dx = \int_a^b f\phi\,dx.$$

$$\Rightarrow -\int_a^b u'\phi'\,dx + \int_a^b u\phi\,dx = \int_a^b f\phi\,dx \quad \text{—(**)}$$

$$\Rightarrow \int_a^b u\phi''\,dx + \int_a^b u\phi\,dx = \int_a^b f\phi\,dx \quad \text{—(***)}$$

(**) If $u \in C^1[a,b]$, $f \in C[a,b]$ then (**) is meaningful.

(***) If $u \in C[a,b]$ $f \in C[a,b]$ then (***) meaningful.

(**) If $u \in L^1(a,b)$, $u' \in L^1(a,b)$ $f \in L^1(a,b)$ is meaningful.

If $u \in C^2[a,b]$, $u(a) = u(b) = 0$ and (**) is satisfied then u is a soln of (*).

$$\int_a^b u''\phi\,dx - u'\phi\Big|_a^b + \int_a^b u\phi\,dx = \int_a^b f\phi\,dx.$$

$$\int_a^b (u'' + u - f)\phi\,dx = 0.$$

$\Rightarrow u'' + u = f$ a.e.

$\because$ u & f are continuous

$u'' + u = f$.

- What if $u \notin C^1[a,b]$?

how we show u is a soln of (*)?

let $I = (a,b)$ (can be unbdd)

$C_0^\infty(I)$

$1 \le p < \infty$.

$$W^{1,p}(I) = \left\{ u \in L^p(I) ;\ \exists\, g \in L^p(I) \ni \int_I u\phi'\,dx = -\int_I g\phi\,dx,\ \forall \phi \in C_0^\infty(I) \right\}$$

(+) —

If $u \in W^{1,p}(I)$ & $\exists g$ satisfying (*) then $u' = g$.

Weak derivative.

$W^{1,p}$ is a Banach space.

$$\|u\|_{W^{1,p}(I)} = \left[\|u\|^p_{L^p(I)} + \|u'\|^p_{L^p(I)}\right]^{1/p}.$$

$p=2$, $W^{1,p}(I) = H^1(I)$

is a Hilbert space with ip

$$(u,v)_{H^1(I)} = (u,v)_{L^2(I)} + (u',v')_{L^2(I)}$$

$$\|u\|_{H^1} = \left(\|u\|^2_{L^2} + \|u'\|^2_{L^2}\right)^{1/2}$$

- $\Omega \subset \mathbb{R}^n$

$$W^{1,p}(\Omega) = \{u \in L^p(\Omega) ; \exists g_1, g_2, \cdots, g_n\}$$

$$\ni \int_\Omega u \frac{\partial \phi}{\partial x_i} = -\int_\Omega g_i \phi \, dx.$$

$\forall \phi \in C_0^\infty(\Omega)$ $i = 1, 2, \cdots, n$

Norm on $W^{1,p}(\Omega)$

$$\|u\|_{W^{1,p}(\Omega)} = \left(\|u\|^p_{L^p} + \sum_{i=1}^n \left\|\frac{\partial u}{\partial x_i}\right\|^p_{L^p}\right)^{1/p}$$

$p=2$ $H^1(\Omega)$ $W^{1,2}(\Omega)$

$$(u,v)_{H^1} = (u,v)_{L^2} + (\nabla u, \nabla v)_{L^2}$$

- How to fix f^ns on $\partial\Omega$ when they are from $L^p(\Omega)$?

<u>Trace of functions</u>

Ω is a Lipschitz domain

$\exists$ a linear operator

$$\gamma : H^1(\Omega) \longrightarrow L^2(\partial\Omega)$$

$\ni$ 1) $\gamma u = u|_{\partial\Omega}$ if $u \in C^1(\bar{\Omega})$

2) $\|\gamma u\|_{L^2(\partial\Omega)} \le C(p,\Omega) \|u\|_{H^1(\Omega)}$

3) Range(γ) is dense in $L^2(\partial\Omega)$.

- γ is NOT surjective.

$H_0^1(\Omega)$, $\Omega \subset \mathbb{R}^n$.

$$W_0^{1,p}(\Omega) = \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{W^{1,p}(\Omega)}}$$

$$H_0^1(\Omega) = \{u \in H^1(\Omega) : u|_{\partial\Omega} = 0\}$$

$W^{-1,p}(\Omega)$.

Set of all linear f^nals

$F : W_0^{1,p}(\Omega) \longrightarrow \mathbb{R}$.

$$W^{-1,p}(\Omega) = W_0^{1,p}(\Omega)$$

$$H^{-1}(\Omega) = (H_0^1(\Omega))^*$$

$F : H_0^1(\Omega) \longrightarrow \mathbb{R}$.

$$\|F\|_{H^{-1}(\Omega)} = \sup\{|\langle F, v\rangle| ; \|v\|_{H_0^1} \le 1\}$$

$$= \sup\left\{\frac{|\langle F, v\rangle|}{\|v\|_{H_0^1(\Omega)}}, v \ne 0\right\}$$

$\Rightarrow |\langle F, v\rangle| \le \|F\|_{H^{-1}(\Omega)} \|v\|_{H_0^1(\Omega)}$

$\downarrow$
duality pairing

H^{-1} is indeed a collection of distributions.

eg: $f \in L^2(\Omega)$.

$\langle F_f, v\rangle = \int_\Omega f v \, dx \quad v \in H_0^1(\Omega)$

$F_f \in H^{-1}(\Omega)$. (Prove)

F_f is linear ~~on~~ $H_0^1(\Omega)$

Enough to show F_f is bdd.

$$|\langle F_f, v\rangle| = \left|\int_\Omega f v \, dx\right|$$
$$\le \|f\|_{L^2} \|v\|_{L^2}$$
$$\le \|f\|_{L^2} \|v\|_{H_0^1}$$

$\|F_f\|_{H^{-1}(\Omega)} \le \|f\|_{L^2}$

by taking sup $|\langle F_f, v\rangle|$ for $\forall v \ni \|v\|_{H_0^1} \le 1$.

• claim H^{-1} is a subspace

$\mathcal{D}'(\Omega) = C_0^\infty(\Omega)^*$

$C_0^\infty(\Omega)$ is dense in $H_0^1(\Omega)$

$$H^{-1}(\Omega) = H_0^1(\Omega)^* \subset C_0^\infty(\Omega)^* = \mathcal{D}'(\Omega)$$

$u, v \in L^2(\Omega)$

$(u, v) = \int_\Omega u v \, dx$.

$H_0^1(\Omega) \subset L^2(\Omega) = (L^2(\Omega))^* \subset H^{-1}(\Omega)$

$f \in L^2(\Omega) \subset H^{-1}(\Omega)$.

$v \in H_0^1(\Omega)$

$$\langle f, v\rangle_{H^{-1}\times H_0^1} = \int_\Omega f v \, dx = (f, v)_{L^2(\Omega)}$$

▶ Dual of H^1 ?

23/12/19

Poisson Eqn.

$$\begin{cases} -\Delta u = f & \text{in } \Omega \subset \mathbb{R}^n \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

$$-\int -\Delta u \, \phi \, dx = \int_\Omega \nabla u \cdot \nabla \phi \, dx - \int_{\partial\Omega} \frac{\partial v}{\partial \nu} \phi \, dx. \quad \to 0$$

(I) $\Rightarrow \int_\Omega \nabla u \cdot \nabla \phi \, dx = \int_\Omega f \phi \, dx \quad \forall \phi \in C_0^\infty(\Omega)$

(this is true for $u \in C^1(\Omega)$)

$\Rightarrow -\int_\Omega u \Delta\phi \, dx + \int_{\partial\Omega} u \frac{\partial \phi}{\partial \nu} ds = \int_\Omega f\phi \, dx$

$\to 0 \because u = 0, \partial\Omega$

(II) $\Rightarrow -\int_\Omega u \Delta \phi \, dx = \int_\Omega f \phi \, dx$.

(this is true for $u \in C(\Omega)$)

(I) ~~to~~ ~~is~~ meaningful for $u \in H_0^1(\Omega)$.

(II) $u \in L^2(\Omega)$, $u = 0$.

<u>Def</u> Let $\Omega \subset \mathbb{R}^n$ is a bdd domain. Then a f^n $u : \Omega \to \mathbb{R}$ is a weak soln. of the Poisson equ

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

if

1) $u \in H_0^1(\Omega)$

2) $\int_\Omega \nabla u \cdot \nabla \phi \, dx = \int_\Omega f \phi \, dx$

$\forall \phi \in H_0^1(\Omega)$
($\because C_0^\infty(\Omega)$ is dense in $H_0^1(\Omega)$).

1) Variational Method.

<u>Gateaux derivative</u> (G.D)

Let X and Y be Banach space. (# locally convex space also sufficient). Let $F : X \to Y$. F is said to be G.D. at $u \in X$ in the direction $k \in X$ if

$$\delta F(u;k) = \lim_{\varepsilon \to 0} \frac{F(u+\varepsilon k) - F(u)}{\varepsilon}$$

$$= \frac{d}{d\varepsilon} F(u+\varepsilon k) \Big|_{\varepsilon = 0}.$$

If such limit exists ~~for~~ $\forall k \in X$ then F is G.D,

$$F'(u)k = \delta(F : k)$$

Eg: H - Hilbert space.

$F(u) = \|u\|_H^2$ $\quad F : H \to \mathbb{R}$.

$$\delta F(u:k) = \lim_{\varepsilon \to 0} \frac{\|u+\varepsilon k\|^2 - \|u\|^2}{\varepsilon}$$

$$= \lim_{\varepsilon \to 0} \frac{\|u\|^2 + \varepsilon^2\|k\|^2 + 2\varepsilon(u,k) - \|u\|^2}{\varepsilon}$$

$$= 2(u,k) \quad \forall k \in H.$$

$$F'(u)k = \delta F(u;k)$$
$$= 2(u,k) \quad \forall k \in H$$

$\Rightarrow F'(u) = 2u$.

$[\, F'(u) : H \to \mathbb{R}\,\}$.
$F'(u)k = 2(u,k)\,]$

Existence of directional derivative $\not\Rightarrow$ even the continuity of f.

<u>Frechet derivative</u> (F.D)

Let X and Y be 2 Banach Spaces. $F : X \to Y$ is said to be F.D at $u \in X$ if $\exists$ bdd linear operator $A : X \to Y$ $\ni$

$$\lim_{h \to 0} \frac{\|F(u+h) - F(u) - Ah\|_Y}{\|h\|_X} = 0 \text{ exists}$$

then $F'(u)h = Ah$.
Also denoted as $DF(u)h = Ah$.

$F'(u)$ (in G.D) need not be a bdd linear but A (in F.D) is.

If $F'(u)$ is bdd linear then G.D & F.D coincides.

__Ex.__ $f \in H^{-1}(\Omega)$, $\Omega \subset \mathbb{R}^n$

Define,

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega fu\, dx.$$

$J : H_0^1(\Omega) \longrightarrow \mathbb{R}.$

__Lemma__

J is differentiable in $H_0^1(\Omega)$.

$$DJ(u)h = \int_\Omega \nabla u \cdot \nabla h\, dx - \int_\Omega fh\, dx$$

$h \in H_0^1(\Omega)$

—

$u \in H_0^1(\Omega)$ $h \in H_0^1(\Omega)$

$J(u+h) - J(u)$

$$= \frac{1}{2}\int_\Omega |\nabla(u+h)|^2 - |\nabla u|^2 dx - \int_\Omega f(u+h) + \int_\Omega fu$$

$$= \frac{1}{2}\int_\Omega |\nabla h|^2 dx + \int_\Omega \nabla u \cdot \nabla h\, dx - \int_\Omega fh\, dx.$$

Define $A : H_0^1(\Omega) \longrightarrow \mathbb{R}$,

$$A(u)h = \int_\Omega \nabla u \cdot \nabla h\, dx - \int_\Omega fh\, dx$$

A is a bdd linear operator

$A(h_1+h_2) = Ah_1 + Ah_2$ $h_1, h_2 \in H_0^1(\Omega)$

$$|Ah| \le \left|\int_\Omega \nabla u \cdot \nabla h\, dx\right| + \left|\int_\Omega fh\, dx\right|$$

$$\le \|\nabla u\|_{L^2} \|\nabla h\|_{L^2} + \|f\|_{H^{-1}} \|h\|_{H_0^1}.$$

$$\|A\|_{H^{-1}(\Omega)} = \sup\left\{ |A(h)h| \;\; \|h\|_{H_0^1} \le 1 \right\}$$

$$\|A\|_{H^{-1}} \le \sup_{\|h\|_{H_0^1} \le 1} \left\{ \|\nabla u\| \underbrace{\|\nabla h\|_{L^2}}_{\le \|h\|_{H_0^1}} + \|f\|_{H^{-1}} \|h\|_{H_0^1} \right\}$$

$$\le \|\nabla u\|_{L^2} + \|f\|_{H^{-1}} < \infty$$

A is bdd linear

$$\lim_{h \to 0} \frac{|J(u+h) - J(u) - Ah|}{\|h\|_{H_0^1}}$$

$$\le \lim_{h \to 0} \frac{1}{2}\int_\Omega |\nabla h|^2 dx.$$

$$\le \lim_{h \to 0} \frac{1}{2}\|h\|_{H_0^1} = 0.$$

- Let X be a Banach space.

$J : X \to \mathbb{R}$ be diff at $x \in X$ and suppose J attains min. at x

$h \in X$

$F(\varepsilon) = J(x + \varepsilon h)$

$$0 = F'(0) = \frac{d}{d\varepsilon} J(x+\varepsilon h)\Big|_{\varepsilon=0}.$$

$\Rightarrow \delta J(x : h) = 0$

J is diff. at $x \Rightarrow$

$DJ(x)h = 0.$

Theorem

Suppose $J : H_0^1(\Omega) \longrightarrow \mathbb{R}$.

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \int_\Omega fu\, dx$$

attains minimum at $u \in H_0^1(\Omega)$. Then u is a weak soln of poisson eqn.

Proof

$DJ(u)h = 0 \quad \forall h \in H_0^1(\Omega)$

$\because J$ attains min at u.

$$\Rightarrow DJ(u)(h) = \int_\Omega \nabla u \cdot \nabla h\, dx - \int_\Omega fh\, dx \quad \forall h \in H_0^1(\Omega)$$

$$\Rightarrow \int_\Omega \nabla u \cdot \nabla h\, dx = \int_\Omega fh\, dx \quad \forall h \in H_0^1(\Omega)$$

$\Rightarrow u$ is a soln. of Poisson eqn.

(Existence of min. at u ?)

1) J is bdd from below in $H_0^1(\Omega)$. Then $\exists$ a minimizer for J.

2) Let $\{u_n\} \in H_0^1$ be the minimizer of J.

3) $u_n \longrightarrow u \in H_0^1(\Omega)$.

- $A_{n\times n}$ be a symmetric +ve definite matrix $b \in \mathbb{R}$.

Find $y \in \mathbb{R}^n \ni Ay = b$.

$J : \mathbb{R}^n \longrightarrow \mathbb{R}$.

$$J(y) = \frac{1}{2}(Ay, y) - (b, y) \quad y \in \mathbb{R}^n.$$

Suppose J attains min. at $x \in \mathbb{R}^n$.

$F(\varepsilon) = J(x+\varepsilon h) \quad h \in \mathbb{R}^n$

$$0 = F'(0) = \frac{d}{d\varepsilon} J(x+\varepsilon h)\Big|_{\varepsilon=0}$$

$$= \lim_{\varepsilon\to 0} \frac{J(x+\varepsilon h) - J(x)}{\varepsilon}$$

$$= \lim_{\varepsilon\to 0} \frac{\frac{1}{2}(A(x+\varepsilon h), x+\varepsilon h) - (b, x+\varepsilon h) - \frac{1}{2}(Ax, x) + (b, x)}{\varepsilon}$$

$$= \lim_{\varepsilon\to 0} \frac{\frac{1}{2}\varepsilon^2 (Ah, h) + \frac{1}{2}\varepsilon(Ax, h) - \varepsilon(b, h)}{\varepsilon}$$

$$= (Ah, x) - (b, h) = 0.$$

$$\Rightarrow (h, Ax) - (b, h) = 0$$

$$(Ax - b, h) = 0 \quad \forall h \in \mathbb{R}^n$$

$$\Rightarrow Ax = b.$$

Thm

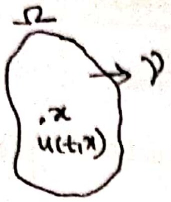
$J(u)$ attains min. at $u \in H_0^1(\Omega)$ iff u is a weak soln of poisson eqn.

~~Going Home~~

KRR.

Diffusion.

Total amount



$= \int_\Omega u(t,x)\,dx$

Flux $= \int_{\partial\Omega} F\cdot\nu \, d\sigma$

Balance

$$\frac{d}{dt}\int_\Omega u(t,x)\,dx = -\int_{\partial\Omega} F\cdot\nu\, d\sigma$$

$$= -\int_\Omega \nabla\cdot F\, dx$$

$$\int_\Omega (\partial_t u + \nabla\cdot F)\,dx = 0.$$

$$\Rightarrow \partial_t u + \nabla\cdot F = 0 \quad \text{---}(1)$$

Conservation law.

$$F = F(t, x, u, Du, D^2u, \cdots)$$

Fourier law.

$$F = -\alpha \nabla u.$$

substituting in (1)

$$\partial_t u + \nabla\cdot(-\alpha\nabla u) = 0$$

$$\Rightarrow \partial_t u = \alpha\Delta u.$$

Heat eqn.

IBVP

$\Omega \subset \mathbb{R}^N$ bdd open set.

$$\begin{cases} \partial_t u - \Delta u = f & \text{in } (0,T)\times\Omega \\ u = g & \text{on } \{0\}\times\Omega \\ u = h & \text{on } (0,T)\times\Gamma \end{cases}$$

$$(\mathrm{I}) - \begin{cases} \partial_t u - \Delta u = f & \text{in } (0,\infty)\times\mathbb{R}^n \\ u = g & \text{on } \{0\}\times\Omega. \end{cases}$$

Cauchy problem.

Non-homo. PDE + Non. homo I.C.

Non-homo PDE + Hom. I.C.

Hom. PDE + Non hom IC

$$\mathrm{II} - \begin{cases} \partial_t u_1 - \Delta u_1 = 0 & \text{in } (0,\infty)\times\mathbb{R}^N \\ u_1 = g & \text{on } \{0\}\times\mathbb{R}^N \end{cases}$$

+

$$\mathrm{III} - \begin{cases} \partial_t u_2 - \Delta u_2 = f & \text{in } (0,\infty)\times\mathbb{R}^N \\ u_2 = 0 & \text{on } \{0\}\times\mathbb{R}^N. \end{cases}$$

If u_1 is a soln. of II and u_2 is a soln of III then u_1+u_2 is a soln ~~of~~ I.

IVP

$$\partial_t u - \Delta u = 0 \quad \text{in } (0,\infty)\times\mathbb{R}^N$$

$$u = g \quad \text{on } \{0\}\times\mathbb{R}^N$$

Heat operator: $\partial_t - \Delta$

Consider the 1D case
$\partial_t u - \partial_x^2 u = 0$.

<u>observation</u> if $\lambda \in \mathbb{R}$ define

$u_\lambda(t,x) := u(\lambda^2 t, \lambda x)$

S.T. u_λ solves the heat eqn.

(let $\tau = \lambda^2 t$, $\xi = \lambda x$ and use chain rule).

$$\frac{\xi^2}{\tau} = \frac{\lambda^2 x^2}{\lambda^2 t} = \frac{x^2}{t}.$$

Look for $u(t,x) = v(\eta)$ where

$\eta = \frac{x^2}{t}$.

$$\partial_t u(t,x) = v'(\eta)\left(-\frac{x^2}{t^2}\right)$$

$$\partial_x u(t,x) = v'(\eta)\left(\frac{2x}{t}\right)$$

$$\partial_x^2 u(t,x) = v''(\eta)\frac{4x^2}{t^2} + \frac{2v'(\eta)}{t}.$$

$$u(t,x) = \frac{1}{(4\pi t)^{1/2}} e^{-x^2/4t}$$

If u_1 solves $\partial_t u_1 - \partial_{x_1}^2 u_1 = 0$

$\vdots$

u_N solves $\partial_t u_N - \partial_{x_N}^2 u_N = 0$

then

$u(t,x) = u_1(t,x_1)\, u_2(t,x_2) \cdots u_N(t,x_N)$

is a soln of II ~~bea~~

(check $u(t,x)$)

$$u(t,x) = \frac{1}{(4\pi t)^{N/2}} e^{-\frac{\|x\|^2}{4t}} \quad t \neq 0.$$

<u>Definition</u>

The fundamental soln of the heat eqn. is defined by

$$\Phi(t,x) := \frac{1}{(4\pi t)^{N/2}} e^{-\frac{\|x\|^2}{t}} \quad (0,\infty)\times\mathbb{R}^N$$

- By construction Φ solves heat eqn.

  $\partial_t \Phi - \Delta \Phi = 0$ in $(0,\infty)\times\mathbb{R}^N$

- Φ and its derivatives remain bdd in $[\delta,\infty)\times\mathbb{R}^N$ $\delta > 0$.

- $\int_{\mathbb{R}^N} \Phi(t,x)\,dx = 1$.

<u>Solution of the Cauchy problem</u>

$$(*)\begin{cases} \partial_t u - \Delta u = 0 \text{ in } (0,\infty)\times\mathbb{R}^N \\ u = g \text{ on } \{0\}\times\mathbb{R}^N \end{cases}$$

<u>Def</u>

A fn $u \in \mathcal{C}^2_1((0,\infty)\times\mathbb{R}^N)$ is called a strong/classical soln to $(*)$ if

$\partial_t u(t,x) - \Delta u(t,x) = 0 \quad \forall (t,x) \in (0,\infty)\times\mathbb{R}^N$

$u(0,x) = g(x) \quad \forall x \in \mathbb{R}^N$.

Theorem

Let $g \in \mathcal{C}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$

and let u be defined by

$$u(t,x) := \int_{\mathbb{R}^N} \Phi(t, x-y)\, g(y)\,dy \qquad t>0,\ x \in \mathbb{R}^N$$

Then the following holds.

i) $u \in \mathcal{C}^\infty((0,\infty)\times\mathbb{R}^N) \cap L^\infty((0,\infty)\times\mathbb{R}^N)$

ii) $\partial_t u - \Delta u = 0$ in $(0,\infty)\times\mathbb{R}^N$

iii) For any $x_0 \in \mathbb{R}^N$

$$\lim_{\substack{x \to x_0 \\ t \to 0^+}} u(t,x) = g(x).$$

($\Rightarrow u$ can be continuous extended to $[0,\infty)$)

(Why we need L^∞ on u?)

Proof

$$u(t,x) = \int_{\mathbb{R}^N} \Phi(t, x-y)\, g(y)\,dy$$

$$|u(t,x)| \le \int_{\mathbb{R}^N} \Phi(t,x-y)\,|g(y)|\,dy$$

$$\le \|g\|_{L^\infty(\mathbb{R}^N)} \underbrace{\int_{\mathbb{R}^N} \Phi(t,x-y)\,dy}_{=1 \ (\text{let } x-y=z)}.$$

$$= \|g\|_{L^\infty(\mathbb{R}^N)}$$

$$\Rightarrow u \in L^\infty((0,\infty)\times\mathbb{R}^N)$$

$$\partial_t u(t,x) = \lim_{k\to 0} \frac{u(t+k,x)-u(t,x)}{k}.$$

$$= \lim_{k\to 0} \int_{\mathbb{R}^N} \frac{\Phi(t+k,x-y)-\Phi(t,x-y)}{k}\, g(y)\,dy$$

$$= \lim_{k\to 0} \int_{\mathbb{R}^N} \partial_t \Phi(t+\theta k, x-y)\, g(y)\,dy$$

by MVT (on t variable)

(Derivative of Φ are bdd)

by DCT

$$= \int_{\mathbb{R}^N} \partial_t \Phi(t, x-y)\, g(y)\,dy.$$

$$= \partial_t \Phi * g(t,x).$$

$\because$ convolution is continuous

$\partial_t u$ is continuous.

$$\partial^\alpha u(t,x) = (\partial^\alpha \Phi * g)(t,x)$$

$\alpha \to$ multi index.

$$\Rightarrow u \in \mathcal{C}^\infty((0,\infty)\times\mathbb{R}^N)$$

$$\partial_t u(t,x) - \Delta u(t,x)$$

$$= \int_{\mathbb{R}^N} [\partial_t \phi(t,x-y) - \Delta\Phi(t,x-y)]\, g(y)\,dy$$

$$= 0.$$

iii) Let $x_0 \in \mathbb{R}^N$ and let $\varepsilon > 0$. Using the continuity of g at x_0 $\exists$ a $\delta > 0 \ni$

~~$|g(y) - g(x_0)| < \varepsilon/2$~~

$\forall y \in \mathbb{R}^N$ $\|y - x_0\| < \delta$

$$\Rightarrow |g(y) - g(x_0)| < \varepsilon/2$$

$|u(t,x) - g(x_0)|$

$= \left| \int_{\mathbb{R}^N} \phi(t, x-y)(g(y) - g(x_0))\, dy \right|$

$\leq \int_{\mathbb{R}^N} \phi(t, x-y)\, |g(y) - g(x_0)|\, dy.$

$= \int_{B(x_0,\delta)} \Phi(t,x-y)\,|g(y)-g(x_0)|\,dy$

$+ \int_{\mathbb{R}^N \setminus B(x_0,\delta)} \Phi(t,x-y)\,|g(y)-g(x_0)|\,dy.$

$= I + J.$

$I \leq \varepsilon/2 \int_{B(x_0,\delta)} \Phi(t,x-y)\, dy.$

$\leq \varepsilon/2 \int_{\mathbb{R}^N} \phi(t,x-y)\,dy = \varepsilon/2.$

$J = \int_{\mathbb{R}^N \setminus B(x_0,\delta)} \Phi(t,x-y)\,|g(y)-g(x_0)|\,dy$

$\leq 2\|g\|_{L^\infty(\mathbb{R}^N)} \int_{\mathbb{R}^N \setminus B(x_0,\delta)} \Phi(t,x-y)\,dy.$

let $\|x - x_0\| < \delta/2$ $\quad y \in \mathbb{R}^N \setminus B(x_0,\delta)$

$\|y - x_0\| \leq \|y - x\| + \|x - x_0\|$

$\leq \|y-x\| + \delta/2$

$\leq \|y-x\| + \frac{1}{2}\|y - x_0\|$

$\|y-x\| \geq \frac{1}{2}\|y - x_0\|$

$\Phi(t, x-y) = \frac{1}{(4\pi t)^{N/2}} e^{\frac{-\|x-y\|^2}{4t}}$

$J \leq \frac{c}{t^{N/2}} \int_{\mathbb{R}^N \setminus B(x_0,\delta)} e^{\frac{-\|y-x_0\|^2}{16t}} dy$

$z = \frac{y - x_0}{\sqrt{t}}$

then $J \to 0$ as $t \to 0^+$.

KS.

*RRT

H_0^1 & H^1.

Remarks

let $H = L^2(\Omega)$. FX

suppose $F \in H^* = L^2(\Omega)$

$\exists!$ $L^2(\Omega)$

$F_f(g) = \int_\Omega fg\,dx \;\; \forall g \in L^2.$

- Poincare inequality

let $\Omega \subset \mathbb{R}^n$ be a bdd domain (atleast one direction)

let $u \in H_0^1(\Omega)$ then $\exists C_p \ni$

$\|u\|_{L^2(\Omega)} \leq C_p \|u\|_{H(\Omega)}$

- Is this true in H^1?

Eg. $w \in C_0^1(0,1)$

$\int_0^1 w^2(x)\,dx \leq 4 \int_0^1 (w'(x))^2 dx$

$\|w\|_{L^2(0,1)} \leq 4\|w'\|^2_{L^2(0,1)}$

$$\int_0^1 w^2(x)\,dx$$

$$= w^2 x\Big|_0^1 - \int_0^1 2x w w'(x)\,dx.$$

$w \in C_0^1$ $\;\; \hookrightarrow 0$

$$\leq \sup_{x\in(0,1)} |x| \int_0^1 |ww'|\,dx$$

$$\leq 2\|w\|_{L^2}\|w'\|_{L^2}$$

$$\Rightarrow \|w\|_{L^2} \leq 2\|w'\|_{L^2}.$$

Equivalent norms

X be a Banach space.

$\exists\, C_1, C_2 > 0$.

$$C_1\|\cdot\|_a \leq \|\cdot\|_b \leq C_2\|\cdot\|_a.$$

$(x_n) \to x$ in $\|\cdot\|_a$.

If $\|x_n - x\|_a \to 0$ as $n\to\infty$

then $\|x_n - x\|_b \to 0$ as $n \to \infty$.

Theorem

Let Ω be a bdd domain in $\mathbb{R}^n$. Then $H_0^1(\Omega)$ with

$$(u,v)_0 = \int_\Omega \nabla u\cdot\nabla v\,dx$$

is a Hilbert space, and the induced norm is equivalent to old norm in $H_0^1(\Omega)$.

ie, $(H_0^1, (\cdot,\cdot)_0)$ and $(H_0^1, (\cdot,\cdot))$ are equivalent.

i) S.T. $(\cdot,\cdot)_0$ is ip

$\langle u, v\rangle_0 = 0 \Rightarrow u = 0$

by Poincare inequality.

ii) S.T $\|\cdot\|_0$ and $\|\cdot\|$ are equivalent

$$\|u\|_0 \leq \|u\|_{H^1}$$

$$= \left(\|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2\right)^{1/2}$$

$$\leq \left(C_p\|\nabla u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2\right)^{1/2}$$

$$\leq \sqrt{1+C_p^2}\,\|u\|_0.$$

$$\|u\|_0 \leq \|u\|_{H^1} \leq \sqrt{1+C_p^2}\,\|u\|_0.$$

$\Rightarrow$ Cauchy sequences in $\|\cdot\|_{H^1}$ & Cauchy in $\|\cdot\|_0$.

$\Rightarrow$ ~~H~~ $(H_0^1, (\cdot,\cdot)_0)$ is a Hilbert space.

Theorem

Let Ω be a bdd domain in $\mathbb{R}^n$

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has a unique weak soln for any $f \in L^2(\Omega)$ (or $H^{-1}(\Omega)$)

Proof

Consider $(H_0^1(\Omega), (\cdot,\cdot)_0)$ and $\|\cdot\|_0$ is equivalent to $\|\cdot\|_{H^1}$

For any $f \in H^1(\Omega)$ or $L^2(\Omega)$

$$F_f(v) = \int_\Omega f\, v\, dx \quad v \in H_0^1(\Omega).$$

$$|F_f| \le \|f\|_{L^2} \|v\|_{L^2} \quad \text{by Holder's}$$

$$\le C_p \|f\|_{L^2} \|v\|_0 \quad \text{by Poincare}$$

$$\|F_f\| \le C_p \|f\|_{L^2}$$

By RRT, $\exists!\, u \in H_0^1(\Omega) \ni$

$$F_f(v) = (u, v)_0 \quad \forall v \in H_0^1(\Omega)$$

$$\int_\Omega f v\, dx = F_f(v) = (u, v)_0$$

$$= \int_\Omega \nabla u \nabla v\, dx.$$

General elliptic problem.

$A(x) = \{a_{ij}\}_{n \times n}$ matrix

$b(x) = (b_1(x), b_2(x), \dots, b_n(x))$

$c(x) = (c_1(x), c_2(x), \dots, c_n(x))$

$d(x) \in \mathbb{R}$. $\quad d(x) > 0 \ \forall x$.

$$Lu := -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij} \frac{\partial u}{\partial x_j}\right) + \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(b_i(x)\, u) + \sum_{i=1}^{n} c_i(x) \frac{\partial u}{\partial x_i} + d(x) u = f$$

$$-\operatorname{div}(A(x) \nabla u) + \operatorname{div}(b u) + c \cdot \nabla u + du = f$$

Uniform ellipticity.

let $\theta, M > 0$

$$\theta |\xi|^2 \le \sum_{i,j=1}^{n} a_{ij}(x) \xi_i \xi_j \le M |\xi|^2$$

$\xi \in \mathbb{R}^n$

A is +ve definite

$$\theta |\xi|^2 \le (A\xi, \xi) := \sum_{i,j=1}^{n} a_{ij}(x) \xi_i \xi_j$$

$$(*) \begin{cases} Lu = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

let $v \in H_0^1(\Omega)$

$$-\int_\Omega \frac{\partial}{\partial x_i}\left(a_{ij} \frac{\partial u}{\partial x_j}\right) v\, dx$$

$$= \int_\Omega a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i}\, dx - \underbrace{\int_{\partial\Omega} a_{ij} \frac{\partial u}{\partial x_j} \nu_i \cdot v\, ds}_{\to 0 \ \because\, v = 0 \text{ on } \partial\Omega}$$

$$a(u, v) = \int_\Omega a_{ij}(x) \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i}\, dx - \int_\Omega b_i(x)\, u \frac{\partial v}{\partial x_i} - \int_\Omega \frac{\partial}{\partial x_i}(c_i(x) v)\, u + \int_\Omega d(x)\, u v\, dx = \int f v\, dx$$

Weak soln.

Find $u \in H_0^1(\Omega)$ ∋

$a(u,v) = (f,v) \ \forall v \in H_0^1(\Omega)$.

RRT fails in this case.
($a(u,v)$ is not symmetric)

Define

$$a^*(v,u) = \int_\Omega \sum_{i=1}^n a_{ij}(x) \frac{\partial v}{\partial x_j} \frac{\partial u}{\partial x_i}$$

$$- \int_\Omega \sum_i b_i v \frac{\partial u}{\partial x_i}$$

$$- \int_\Omega \frac{\partial}{\partial x_i} (c_i u) v$$

$$+ \int_\Omega d u v$$

Note that $a(u,v) \neq a^*(v,u)$

let $c_i = b_i = 0 \quad \forall i = 1, 2, \dots, n$

Assume that $a_{ij}(x) = a_{ji}(x)$

ie, A is symmetric.

$$((u,v)) = \int_\Omega \sum_{i,j=1}^n a_{ij}(x) \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i}$$

$$+ \int_\Omega d(x) u v \, dx$$

$$= ((v,u))$$

$$((u,u)) = \int_\Omega \sum_{i,j=1}^n a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} dx$$

$$+ \int_\Omega d |u|^2 dx .$$

$$\geq \theta \int_\Omega \sum_{i=1}^n \left| \frac{\partial u}{\partial x_i} \right|^2 dx$$

$$+ \int_\Omega d_0 |u|^2 dx$$

$$\Rightarrow \theta_1 \int_\Omega |\nabla u|^2 + |u|^2) \quad \left(\theta_1 = \min\{\theta, d_0\} \right.$$

$$= \theta_1 \|u\|_1^2 .$$

$$((u,u)) = \|u\|_0^2 = 0$$

$$\Rightarrow \|u\|_1^2 = 0 \quad \Rightarrow u = 0 .$$

and $\theta_1 \|u\|_1 \leq \|u\|_0$.

Let $d_0 \leq d(x) \leq d_1 < \infty$

$$((u,u)) \leq M \int_\Omega |\nabla u|^2 dx$$

$$+ d_1 \int_\Omega |u|^2 dx .$$

$$\leq M_1 \|u\|_1 \quad \left(M_1 = \max\{M, d_1\} \right.$$

$\Rightarrow \|\cdot\|_1$ & $\|\cdot\|_0$ are equivalent.

KRA

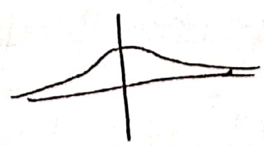
Infinite speed of propagation.

Let $g \in C_c(\mathbb{R}^N)$

eg: $g(x) = 0$ if $\|x\| > 1$.

$g > 0$ in $B(0,1)$.

$u(t,x) > 0 \quad \forall (t,x) \in (0,\infty) \times \mathbb{R}^N$.



at initial stage soln will have compact support. But within no time heat travels. So soln won't have compact support.

Non-homogeneous problem.

$$\partial_t u - \Delta u = f \quad \text{in } (0,\infty)\times\mathbb{R}^N.$$
$$u = 0 \quad \text{on } \{0\}\times\mathbb{R}^N.$$

Duhamel's principle.

$$y' + ay = f \quad \text{in } (0,\infty)$$
$$y(0) = y_0$$

$$e^{at}y' + a e^{at} y = e^{at} f.$$

$$\frac{d}{dt}(e^{at}y) = e^{at} f.$$

$$e^{at}y - y_0 = \int_0^t e^{a\tau} f \, d\tau$$

$$\Rightarrow y(t) = e^{-at} y_0 + \int_0^t e^{-a(t-\tau)} f(\tau)d\tau$$

soln. of the homogeneous problem

$$y' + ay = 0 \text{ in } (0,\infty)$$
$$y(0) = y_0.$$

$$y(t) = e^{-at} y_0 = S(t)y_0$$

For $t>0$

$$S(t) : \mathbb{R} \longrightarrow \mathbb{R}$$
$$y_0 \longmapsto y(t)$$

S is called solution operator.

$$y(t) = S(t)y_0 + \int_0^t S(t-\tau) f(\tau)\, d\tau$$

In the case of heat eqn.

$$u(t,\cdot) = \int_{\mathbb{R}^N} \Phi(t,\cdot-y) g(y)\,dy$$

$$S(t)\, g = u(t,\cdot) \qquad S = \phi$$

$$u(t,\cdot) = \int_0^t S(t-\tau) f(\tau,\cdot)\, d\tau.$$
$$= \int_0^t \int_{\mathbb{R}^N} \Phi(t-\tau, x-y) f(\tau,y)\,dy\,d\tau$$

Soln operator method
– linear case works.
(Semigroup)

Mean value Property

If $u \in \mathcal{C}^2(\Omega)$ is harmonic then

$$u(x) = \frac{1}{|S(x,r)|}\int_{S(x,r)} u(y)\,d\sigma(y)$$

$S \to$ sphere

spheres are the level sets of Φ

Def^n (Heat ball)

The heat ball of radius r and "centre" at (t,x) is the set

$$E(t,x;r) := \left\{ (\tau,\xi) : \tau \le t,\ \Phi(t-\tau, x-\xi) \ge \frac{1}{r^N} \right\}$$

~~$\Phi(t-\tau)$~~

$$\Phi(t-\tau, x-\xi) = \frac{1}{(4\pi(t-\tau))^{N/2}} e^{-\frac{\|x-\xi\|^2}{4(t-\tau)}}$$

exponential has the max. value when $\xi = x$

$$\Phi(t-\tau, 0) = \frac{1}{(4\pi(t-\tau))^{N/2}}$$

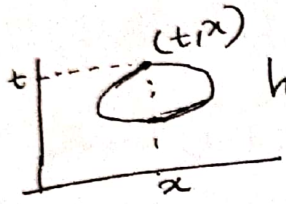
When

$$\Phi(t-\tau,0) = \frac{1}{r^N}$$

$$\Rightarrow \quad \tau = t - \frac{r^2}{4\pi}$$

When $t - \frac{r^2}{4\pi} < \tau < t$

$$\frac{1}{(4\pi(t-\tau))^{N/2}} e^{\frac{-\|x-\xi\|^2}{4(t-\tau)}} = \frac{1}{r^N}$$



heat ball

<u>Parabolic cylinder.</u>

Let $\Omega \subset \mathbb{R}^N$ be a bdd open set and $T>0$. The parabolic cylinder is the set

$$\Omega_T := (0,T] \times \Omega$$

Parabolic bdry,

$$\Gamma_T = \overline{\Omega}_T \setminus \Omega_T$$

<u>Theorem (MVP)</u>

Let $u \in \mathcal{C}_1^2(\Omega)$ be a soln of the heat eqn. Then

$$u(t,x) = \frac{1}{4r^N} \iint_{E(t,x;r)} u(\tau,\xi) \frac{\|x-\xi\|^2}{(t-\tau)^2} \, d\xi \, d\tau$$

<u>Maximum Principle.</u>

Let $\Omega \subset \mathbb{R}^N$ be a bdd open set in $\mathbb{R}^N$ and let $u \in \mathcal{C}_1^2(\overline{\Omega}_T)$ be a soln of the heat eqn. Then the following holds.

i) Weak max. principle.

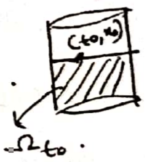
$$\max_{\overline{\Omega}_T} u = \max_{\Gamma_T} u.$$

ii) Strong max. principle.

If Ω is connected and if $\exists$ a point $(t_0,x_0) \in \Omega_T \ni$

$u(t_0,x_0) = \max_{\overline{\Omega}_T} u$ then

$u =$ const. in $\overline{\Omega}_{t_0}$.



<u>Uniqueness of solns to IBVP.</u>

Let $g \in \mathcal{C}(\Gamma_T)$ $f \in \mathcal{C}(\Omega_T)$ then the IBVP,

$$\partial_t u - \Delta u = f \quad \text{in } \Omega_T$$

$$u = g \quad \text{on } \Gamma_T.$$

has a unique soln.

* <u>Uniqueness for unbdd domains</u>

$$\partial_t u - \Delta u = 0 \quad \text{in } (0,T) \times \mathbb{R}^N$$

$$u = g \quad \text{on } \{0\} \times \mathbb{R}^N.$$

Suppose u is a soln satisfying an exponential decay

ie, $u(t,x) \leq A e^{a\|x\|^2}$

a, A are const

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Then

$$\sup_{[0,T]\times\mathbb{R}^N} u = \sup_{\mathbb{R}^N} u(0,\cdot) = \sup_{\mathbb{R}^N} g$$

24/12/19

Wave equation

IVP for the wave equation.

$$(1)\begin{cases} \partial_t^2 u - c^2 \Delta u = 0 & \text{in } (0,\infty)\times\mathbb{R}^N \\ u = g & \text{on } \{0\}\times\mathbb{R}^N \\ \partial_t u = h & \text{on } \{0\}\times\mathbb{R}^N \end{cases}$$

(eqn. can be reduced to 1st order in time by taking $u_t = v$ in that case we need to prescribe $v(0)$. ie why we prescribe $\partial_t u$ also)

Def

A strong / classical soln of the IVP is a fn ~~$u\in C^2(0,\infty$~~ $u \in C^2((0,\infty)\times\mathbb{R}^N)\cap C^1([0,\infty)\times\mathbb{R}^N)$ which satisfies (1) pointwise.

One dimensional case

$$\partial_t^2 u - c^2\partial_x^2 u = 0 \quad \text{in } (0,\infty)\times\mathbb{R}$$
$$u = g \quad \text{on } \{0\}\times\mathbb{R}$$
$$\partial_t u = h \quad \text{on } \{0\}\times\mathbb{R}.$$

change of variable

$$(t,x) \longleftrightarrow (\tau,\xi)$$

$$\partial_t u = \partial_\tau u\, \partial_t\tau + \partial_\xi u\, \partial_t\xi$$

$$\partial_t^2 u = \partial_\tau^2 u(\partial_t\tau)^2 + 2\partial_{\tau\xi}^2 u(\partial_t\tau)(\partial_t\xi) + \partial_\xi^2 u\,(\partial_t\xi)^2.$$

$$\partial_t^2 u - c^2\partial_x^2 u$$
$$= \partial_\tau^2 u\{(\partial_t\tau)^2 - c^2(\partial_x\tau)^2\}$$
$$+ 2\partial_{\tau\xi}^2 u\{\partial_t\tau\,\partial_t\xi - c^2\partial_x\tau\,\partial_x\xi\}$$
$$+ \partial_\xi^2 u\{(\partial_t\xi)^2 - c^2(\partial_x(\xi))^2\}$$

Choose $\tau(t,x)$ and $\xi(t,x)$ in such a way that

$$(\partial_t\tau)^2 - c^2(\partial_x\tau)^2 = 0$$
$$(\partial_t\xi)^2 - c^2(\partial_x\xi)^2 = 0.$$

$$(\partial_t\tau + c\,\partial_x\tau)(\partial_t\tau - c\,\partial_x\tau) = 0$$

Choose $\tau \ni$ $\partial_t\tau + c\,\partial_x\tau = 0$ and $\xi \ni$ $\partial_t\xi - c\,\partial_x\xi = 0.$

Choose

$$\tau(x,t) = x - ct$$
$$\xi(x,t) = x + ct.$$

The wave equation reduces to $\partial_{\tau\xi}^2 u = 0.$

$$\partial_\xi(\partial_\tau u) = 0$$

$\Rightarrow \partial_\tau u = f(\tau)$, f is arbitrary.

$u(\eta,\xi) = F(\eta) + G(\xi)$

F & G are arbitrary.

Finally,

$u(t,x) = F(x-ct) + G(x+ct)$

$u(0,x) = g(x)$.

$\Rightarrow g(x) = F(x) + G(x)$

$\partial_t u(t,x) = (-c)F'(x-ct) + c\,G'(x+ct)$ —(2)

$\partial_t u(0,x) = h(x)$.

$h(x) = -cF'(x) + cG'(x)$

$-c(F(x)-F(0)) + c(G(x)-G(0)) = \int_0^x h(s)\,ds$.

$-cF(x) + cG(x) = -cF(0) + cG(0) + \int_0^x h(s)\,ds$

$cF(x) + cG(x) = \cancel{-cF(0)+cG(0)}\; c\,g(x)$

Adding the above 2 eqns,

$2cG(x) = -cF(0) + cG(0) + cg(x) + \int_0^x h(s)\,ds$.

$2cF(x) = cg(x) - \int_0^x h(s)\,ds + cF(0) - cG(0)$.

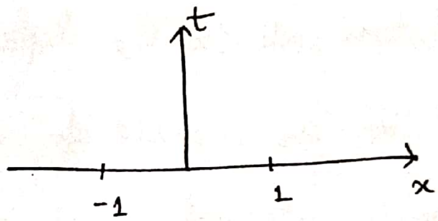
$$u(t,x) = \frac{1}{2}\left(g(x-ct) + g(x+ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} h(s)\,ds \quad \text{—}(*)$$

If $g \in C^2(\mathbb{R})$ and $h \in C^1(\mathbb{R})$ then $u \in C^2((0,\infty)\times\mathbb{R})$

s.T. $u \in C^1([0,\infty)\times\mathbb{R}^N)$. (using (2))

D'Alembert formula $(*)$ → classical soln.

Assume that $g, h = 0$ if $|x|>1$.



let $c = 1$.

$$u(1,x) = \frac{1}{2}(g(x-1) + g(x+1)) + \frac{1}{2}\int_{x-1}^{x+1} h(s)\,ds.$$?

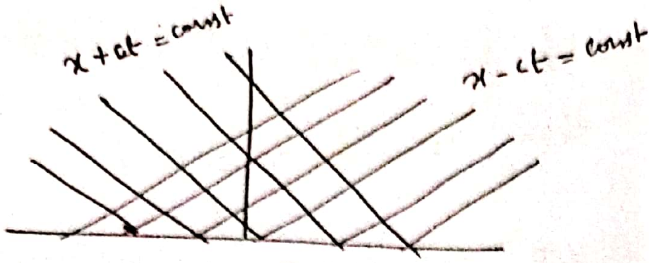
If $-2<x<2$, then $u(1,x) \neq 0$

else $u(1,x) = 0$.

Def.

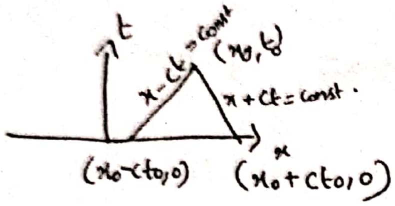
Characteristic.

The family of straightlines $x+ct = \text{const}$ and $x-ct = \text{const}$ are called characteristic curves of the wave eqn.



$u(t,x) = \frac{1}{2}\left[g(x-ct) + g(x+ct)\right]$

$+ \frac{1}{2c}\int_{x-ct}^{x+ct} h(s)\,ds.$



$(x,z)-(x,y)$
$-(y,z)+(y,y)$

The solution at (x_0,t_0) depends only on the value of g & h in the interval $[x_0-ct_0, x_0+ct_0]$

The <u>domain of dependence</u> of a point (x_0,t_0) is the interval $[x_0-ct_0, x_0+ct_0]$.

<u>KS</u>.

<u>Theorem</u>

Let H be a real Hilbert space. Let M be a closed convex subset of H. Then for any $x \in H$, $\exists!\ y \in M$

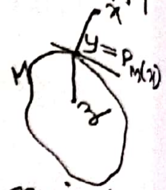
$\ni \quad \|x-y\| = \inf_{z\in M} \|x-z\| \quad \text{—(1)}$

Moreover, $y \in M$ can be characterized by

$(x-y, z-y)_H \le 0 \quad \forall z \in M \quad \text{—(2)}$?

The element $y \in M$, is characterized as a projection of x on M

$y = P_M(x)$.



Infact ① & ② are equivalent.

If ① is true, $z \in M$, $y \in M$

$0 < t < 1 \quad tz + (1-t)y \in M$.

$\|x-y\| \le \|x - tz - (1-t)y\|$

$= \|x-y - t(z-y)\|$

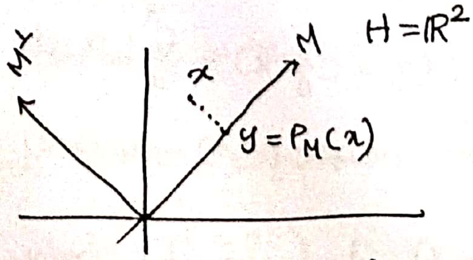
<u>Corollary</u>.

M is a subspace of H.

$(x-y, z') \le 0 \quad \forall z' \in M$

$\Rightarrow (x-y, -z') \le 0 \quad -z' \in M$

$(x-y, z) = 0 \quad \forall z \in M.$



$(x-y, y) = 0$

$(x-y, z) = 0$

$\forall z \in M$

$H = M \oplus M^{\perp}$.

Proposition

$P_M : H \longrightarrow M$ is continuous.

$x_1, x_2 \in H$

$\| P_M(x_1) - P_M(x_2) \|_H \leq \| x_1 - x_2 \|$

Let V_1, V_2 be two linear spaces.

A map $a(\cdot,\cdot) : V_1 \times V_2 \longrightarrow \mathbb{R}$ bilinear if

1) $\forall y \in V_2$ $x \longmapsto a(x,y)$ is linear in V_1

2) $\forall x \in V_1$ $y \longmapsto a(x,y)$ is linear in V_2.

$V = V_1 = V_2$ then a is bilinear on V.

eg: $a(u,v) = \int_a^b [p(x) u'(x) v'(x) + q(x) u'(x) v(x) + r(x) u v] \, dx$

p, q, r bdd f^ns on $[a,b]$ is bilinear form on $C^1[a,b]$.

Def

Let H be a Hilbert space

let $a : H \times H \longrightarrow \mathbb{R}$ be a bilinear map.

Then a is continuous on H if $\exists M > 0 \ni$

$$|a(u,v)| \leq M \|u\| \|v\| \quad \forall u, v \in H$$

a is coercive if $\exists c > 0 \ni$

$$a(u,u) \geq c \|u\|_H^2 \quad \forall u \in H.$$

(bdd below by zero - existence of min)

eg: Let A be a $n \times n$ real +ve symmetric matrix.

$$A = \{a_{ij}\}_{1 \leq i,j \leq n}.$$

$u \in \mathbb{R}^n$ $\quad Au = \sum_{j=1}^{n} a_{ij} u_j$

$v \in \mathbb{R}^n$ $\quad (Au, v) = \sum_{i,j=1}^{n} a_{ij} u_j v_i$

$= v^T A u$

$= (v, Au)$

Define $a(u,v) = (Au, v)$

$|a(u,v)| = |(Au, v)|$

$\leq |Au| |v|$

$\leq \underbrace{\|A\|}_{=K} |u| |v|$

$\because A$ is symmetric and +ve definite

$a(u,u) = (Au, u) \geq c|u|^2$.

$c \rightarrow$ least eigenvalue of the

$a(u,v) = (Au, v)$ is continuous and coercive.

<u>Stampacchia theorem</u>

Let H be a Hilbert space and $a : H \times H \longrightarrow \mathbb{R}$ be a continuous coercive bilinear form on H. Let M be a closed convex

* subset of M. Then for any $f \in H (= H^*)$ $\exists!\, u \in M \ni$

$$a(u, v-u) \geq (f, v-u) \ \forall v \in M.$$

<u>Proof</u>.

Let $u \in H$. $v \longmapsto a(u,v)$ is a linear map on H.

$\because$ $v \longmapsto a(u,v)$ is a continuous linear functional on H.

By RRT $\exists$ (denote $Au \in H$)

$\ni$ $a(u,v) = (Au, v) \ \forall v \in H.$

For any $v \in H$

$u \rightarrow Au$ is linear and bdd.

$$|(Au, v)| = |a(u,v)|$$
$$\leq \|A\| \|u\| \|v\|$$

A is coercive.

$$(Au, u) = a(u,u) \geq C\|u\|^2.$$

Find $u \in M \ni$

$$(Au, v-u) \geq (f, v-u) \ \forall v \in M$$

Let $\rho > 0$ be any real no.

$$(\rho f, v-u) \geq (\rho Au, v-u)$$

$$(\rho f - \rho Au + u - u, v-u) \leq 0$$

Let P_M be the projection on H to M

S.T. $u = P_M(\underbrace{\rho f - \rho Au + u}_{\bar{x}})$

then $\exists!\, u \in M \ni$

$$(\bar{x} - u, z-u) \leq 0 \ \forall z \in M.$$

Take $v_1, v_2 \in M$,

$$\|F(v_1) - F(v_2)\|$$
$$= \|P_M(\rho f - \rho A v_1 + v_1) - P_M(\rho f - \rho A v_2 + v_2)\|$$

$$\leq \|v_1 - v_2 - \rho A(v_1 - v_2)\|$$

$\cancel{\leq \|P_M\|}$

$$\|F(v_1) - F(v_2)\|^2$$

$$\leq \|v_1 - v_2\|^2 + \rho^2 \|A(v_1 - v_2)\|^2 - 2\rho(A(v_1 - v_2), v_1 - v_2)$$

&

$$\|A(v_1 - v_2)\|^2 \leq \underbrace{\|A\|^2}_{K^2} \|v_1 - v_2\|^2$$

$(A(v_1-v_2), v_1-v_2) \geq c\|v_1-v_2\|^2$

$\|F(v_1)-F(v_2)\|^2$

$\leq (1+(\rho k)^2 - 2\rho c)\ \|v_1-v_2\|$

We want $\alpha := 1+(\rho k)^2 - 2\rho c < 1$ inorder to get contraction.

Choose $0 < \rho < 2c/k^2$.

$\|F(v_1)-F(v_2)\| \leq \sqrt{\alpha}\ \|v_1-v_2\|$

F is contraction.

F has a unique fixed pt.

$F(v) = v \quad \forall \in M.$

[let $F: M \to M \ni$

$F(v) = P_m(\rho f - \rho A v + v)$]

KRA

ω_N : the volume of unit ball in $\mathbb{R}^N$

eg: $\omega_3 = \frac{4}{3}\pi = |B(0,1)|$

Volume of a ball of radius

$\rho = \omega_N \rho^N = |B(0,\rho)|$

Surface area of the unit sphere $= |S(0,1)|$

$= N\omega_N$

$|S(0,\rho)| = N\omega_N \rho^{N-1}$

Def^n (Spherical mean)

If $g \in \mathcal{C}(\mathbb{R}^N)$ the spherical mean of g, taken over $S(x,\rho)$ $x \in \mathbb{R}^N$, $\rho > 0$ is defined by

$$M_g(\rho, x) := \frac{1}{|S(x,\rho)|} \int_{S(x,\rho)} g(y)\, d\sigma(y)$$

$x \in \mathbb{R}^N$, $\rho > 0$.

Proposition

1) If $g \in \mathcal{C}^k(\mathbb{R}^N)$ then $M_g(\rho,\cdot)$ is $\mathcal{C}^k$ for each $\rho > 0$.

~~Let $x_0 \in \mathbb{R}^N$. For any $\varepsilon > 0$, $\exists$ a $\delta > 0 \ni$ $\forall x \in \mathbb{R}^N$, $\|x - x_0\| < \delta$ $\Rightarrow |g(x) - g(x_0)| < \varepsilon$.~~

$$M_g(\rho, x) = \frac{1}{N\omega_N \rho^{N-1}} \int_{S(0,1)} g(x+\rho z)\rho^{N-1} d\omega(z)$$

$(\because y \in S(x,\rho),\ y = x + \rho z \quad z \in S(0,1))$

$$= \frac{1}{N\omega_N} \int_{S(0,1)} g(x+\rho z)\, d\omega(z)$$

2) The f^n value $g(x)$ can be recovered by taking limit

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$$g(x) = \lim_{\rho \to 0+} M_g(\rho, x)$$

$$M_g(\rho, x) = \frac{1}{N\omega_N} \int_{S(0,1)} g(x+\rho z)\, d\omega(z)$$

$z \longrightarrow -z$

$$M_g(\rho, x) = \frac{1}{N\omega_N} \int_{S(0,1)} g(x-\rho z)\, d\omega(z)$$

$$= M_g(-\rho, x)$$

$$M_g(0, x) = g(x).$$

Euler – Poisson – Darboux eqn.

If $g \in \mathcal{C}^2(\mathbb{R}^N)$ then M_g satisfies the PDE

$$\partial_\rho^2 M_g + \frac{(N-1)}{\rho} \partial_\rho M_g = \Delta M_g .$$

$$M_g = \frac{1}{N\omega_N} \int_{S(0,1)} g(x+\rho z)\, d\omega(z)$$

$\partial_\rho M_g(\rho, x)$ (How to take ∂_ρ inside?)

$$= \frac{1}{N\omega_N} \int_{S(0,1)} \nabla g(x+\rho z) \cdot z \, d\omega(z)$$

let $z \longmapsto y$

$$= \frac{1}{N\omega_N} \int_{S(x,\rho)} \nabla g(y)\, \frac{y-x}{\rho}\, \frac{d\sigma(y)}{\rho^{N-1}}$$

$$= \frac{1}{N\omega_N \rho^{N-1}} \int_{S(x,\rho)} \nabla g(y) \cdot \nu(y)\, d\sigma(y)$$

by divergence thm

$$= \frac{1}{N\omega_N \rho^{N-1}} \int_{B(x,\rho)} \Delta g(y)\, dy$$

by

$$\rho^{N-1} \partial_\rho M_g = \frac{1}{N\omega_N} \int_{B(x,\rho)} \Delta g(y)\, dy$$

$$= \frac{1}{N\omega_N} \int_0^\rho \int_{S(x,r)} \Delta g(y)\, r^{N-1}\, d\sigma\, dr$$

$\because$ we can cover the ball by spheres.

$$= \frac{1}{N\omega_N} \int_0^\rho r^{N-1}\, dr \int_{S(x,r)} \Delta g(y)\, d\sigma$$

$$\partial_\rho\left(\rho^{N-1} \partial_\rho M_g\right) = \Delta_x\left(\rho^{N-1} M_g\right)$$

Solution of the 3-D IVP

$$\left.\begin{aligned} &(1)\quad \partial_t^2 u - \Delta u = 0 \quad \text{in } (0,\infty) \times \mathbb{R}^3 \\ &(2)\quad u = g \quad \text{on } \{0\} \times \mathbb{R}^3 \\ &(3)\quad \partial_t u = h \quad \text{on } \{0\} \times \mathbb{R}^3 \end{aligned}\right\} - \mathrm{I}$$

For each fixed $t \geq 0$ and $x \in \mathbb{R}^3$, define

$$M_u(t, \rho, x) := \frac{1}{N\omega_N} \int_{S(0,1)} u(t, x+\rho z)\, d\omega(z)$$

$$\Delta M_u(t, \rho, x) = \frac{1}{N\omega_N} \int_{S(0,1)} \Delta u(t, x+\rho z)\, d\omega(z)$$

$$= \frac{1}{N\omega_N} \int_{S(0,1)} \partial_t^2 u(t, x+\rho z)\, d\omega(z)$$

$= \partial_t^2 \left(\frac{1}{N\omega_N} \int_{S(0,1)} u(t, x+\rho z)\, d\omega(z) \right)$

$= \partial_t^2 Mu\,(t, \rho, x)$

If u solves (1) the Mu solves

$$\left.\begin{aligned} \partial_t^2 Mu &= \partial_\rho^2 Mu + \frac{2}{\rho}\partial_\rho Mu \quad t>0,\ \rho \in \mathbb{R} \\ Mu\,(0,\rho) &= Mg(\rho) \quad \rho \in \mathbb{R} \\ \partial_t Mu\,(0,\rho) &= M_h(\rho) \quad \rho \in \mathbb{R} \end{aligned}\right\} \text{—II}$$

I → 3D problem converted to II - 1D problem.

converse is also true,

Define $u(t,x) := \lim_{\rho \to 0} Mu(t,\rho,x)$

then ~~sol~~ Mu solves II ⇒ u solves I.

Define $L(t,\rho) := \rho\, M_u(t,\rho)$

$\partial_t L = \rho\, \partial_t Mu$

$\partial_t^2 L = \rho\, \partial_t^2 Mu \cancel{+ 2\partial}$

$\because \partial_\rho L = \rho\, \partial_\rho Mu + Mu.$

$\partial_\rho^2 L = \rho\, \partial_\rho^2 Mu + 2\partial_\rho Mu.$

Since Mu solves

$\partial_t^2 Mu = \partial_\rho^2 Mu + \frac{2}{\rho}\partial_\rho Mu$

$\frac{\partial_t^2 L}{\rho} = \frac{\partial_\rho^2 L}{\rho}$

$\Rightarrow \partial_t^2 L - \partial_\rho^2 L = 0.$

$L(0,\rho) = \rho\, Mu(0,\rho)$

$\partial_t L(0,\rho) = \rho\, Mu(0,\rho)$

$= \rho\, Mh(\rho)$

∴ L solves the IVP,

$$\left.\begin{aligned} \partial_t^2 L - \partial_\rho^2 L &= 0 \quad t>0,\ \rho \in \mathbb{R}. \\ L(0,\rho) &= \rho\, Mg(\rho) \quad \rho \in \mathbb{R} \\ \partial_t L(0,\rho) &= \rho\, M_h(\rho) \quad \rho \in \mathbb{R} \end{aligned}\right\}$$

$$L(t,\rho) = \frac{\left[(\rho+ct)\, Mg(\rho+ct) + (\rho-ct)\, Mg(\rho-ct)\right]}{2} + \frac{1}{2c}\int_{\rho-ct}^{\rho+ct} \eta\, M_h(\eta)\, d\eta$$

$Mu\,(t,\rho,x) = \frac{L(t,\rho)}{\rho}$

Finally $u(t,x) = \lim_{\rho \to 0} Mu(t,\rho,x)$

$= \lim_{\rho \to 0} \frac{L(t,\rho)}{\rho}.$

KS

If $a(\cdot,\cdot)$ is symmetric $u \in M$ minimizes the f^n^al

$J(v) = \frac{1}{2} a(u,v) - (f, v)$

Define

$((u,v)) := a(u,v)$

S.T. $((\cdot,\cdot))$ is an I.P.

$((u,u)) = 0 \Rightarrow u = 0$ by coercivity

$((u,v)) = ((v,u))$ by symmetry of a

$\|u\|_0^2 = ((u,u))$

$c\|u\|^2 \leq a(u,u) = \|u\|_0^2$

$\leq k\|u\|^2$

$\|\cdot\|$ & $\|\cdot\|_0$ are equivalent

$(H, ((\cdot)))$ is a Hilbert space (**)

$a(\cdot,\cdot)$ is a bilinear fnal on H

By R.B.T. $\exists\ \tilde{f} \in H$

$\ni a(\tilde{f}, v) = ((\tilde{f}, v))\ \forall v \in H$

$= (f,v)$ by old innerproduct & $\because f \in H$.

Consider

$\frac{1}{2}\|v - \tilde{f}\|^2$

$= \frac{1}{2}((v-\tilde{f}, v-\tilde{f}))$

$= \frac{1}{2}a(v,v) - \frac{1}{2}a(v,\tilde{f})$
$- \frac{1}{2}a(\tilde{f},v) + \frac{1}{2}a(\tilde{f},\tilde{f})$

$= \frac{1}{2}a(v,v) - (f,v) + \frac{1}{2}\|\tilde{f}\|_0^2$

$= J(v) + \frac{1}{2}\|\tilde{f}\|_0^2$.

Finding min. of $\|v - \tilde{f}\|_0$ $v \in M$; is equal to the inequality

$\exists !\ u \in M \ni$

$((\tilde{f} - u, v - u)) \leq 0\ \forall v \in M$

$\Rightarrow a(\tilde{f} - u, v-u) \leq 0$

$\Rightarrow a(\tilde{f}, v-u) \leq a(u, v-u)$

$\Rightarrow a(u, v-u) \geq (f, v-u)\ \forall v \in M$

let $M = V$ be the subspace of H.

V is convex in H.

by (**)

$a(u,v') \geq (f,v')\ \forall v' \in V$

$a(u,-v') \geq (f,-v')$

$\Rightarrow a(u,v) = (f,v)\ \forall v \in V$

take $V = H$.

$\Rightarrow a(u,v) = (f,v)\ \forall v \in H$.

Lax-Milgram Lemma

Let H be a Hilbert space.

Let $a(\cdot,\cdot)$ be a continuous coercive bilinear form on H.

$(a : H \times H \longrightarrow \mathbb{R})$

Then for any $f \in H (= H^*)$

$\exists !\ u \in H \ni$

$$a(u,v) = (f,v) \quad \forall v \in H$$

Moreover, if $a(\cdot,\cdot)$ is symmetric u solves the minimization problem

$$J(v) = \frac{1}{2} a(v,v) - (f,v) \quad \forall v \in H.$$

eg: $\begin{cases} -u'' + u = f & \text{in } (a,b) \\ u(a) = u(b) = 0 & u' = \frac{du}{dx} \end{cases}$

Let $v \in H_0^1(a,b)$ $\quad f \in L^2(\Omega)$

$$+\int_a^b u'v'\,dx + \int_a^b uv\,dx = \int_a^b fv\,dx.$$

$$\int_a^b u'v' + uv\,dx = \int_a^b fv\,dx.$$

Find $u \in H_0^1(a,b)$ $\forall v \in H_0^1(a,b)$

$a : H_0^1 \times H_0^1 \longrightarrow \mathbb{R}$

$$a(u,v) = \int_a^b (u'v' + uv)\,dx$$

Let $f \in L^2(a,b)$. Find $u \in H_0^1(a,b) \ni$

$$a(u,v) = (f,v) \quad \forall v \in H_0^1.$$

P.T. $a(\cdot,\cdot)$ satisfies

1) bilinearity
2) continuity
3) coercivity on H_0^1

$$|a(u,v)| \leq \int_a^b |u'v'| + |uv|\,dx.$$

$$\leq \|u'\|_{L^2}\|v'\|_{L^2} + \|u\|_{L^2}\|v\|_{L^2}$$

$$\leq \|u'\|_{L^2}\|v'\|_{L^2} + C_p\|u'\|_{L^2}\|v'\|_{L^2}$$

by Poincare

$$= (1 + C_p)\|u'\|_{H_0^1}\|v'\|_{H_0^1}$$

$$a(u,u) = \int_a^b |u'|^2 + |u|^2\,dx.$$

$$= \|u'\|_{L^2}^2 + \|u\|_{L^2}^2$$

$$\geq \|u'\|_{L^2}^2 = \|u\|_{H_0^1}^2.$$

$\Rightarrow$ a satisfies (1), (2) & (3).

$\ldots\ H^{-2}\quad H^{-1}\quad H^0 = L^2\quad H^1\quad H^2\quad H^3\ \ldots$

$u \in H^1 \Rightarrow u' \in L^2$

$u \in L^2 \Rightarrow u' \in H^{-1}$

Define

$$F_f(v) = \int_a^b f v \, dx \quad v \in H_0^1(\Omega)$$

$$\| F_f \|_{H^{-1}} \leq \| f \|_{L^2}$$

$\Rightarrow F_f$ is bilinear f'nal.

By Lax-Milgram lemma,

$\exists !\ u \in H_0^1 \ni a(u,v) = (f,v)$

2) Poisson eqn.

$$\begin{cases} -\alpha \Delta u + a(x) u = f & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = g & \text{in } \partial\Omega \end{cases}$$

→ heat flux in normal direction.

Assume $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ ↳ to say u is classical sol.

let $v \in C^1(\bar{\Omega})$

$$\int_\Omega \alpha \nabla u \nabla v - \int_{\partial\Omega} \alpha \frac{\partial u}{\partial \nu} v \, ds + \int_\Omega a(x) u v \, dx = \int_\Omega f v \, dx$$

$C^1(\bar{\Omega})$ is dense in $H^1(\Omega)$

Define

$a : H^1 \times H^1 \longrightarrow \mathbb{R}$.

$$a(u,v) = \int_\Omega \alpha \nabla u \nabla v + a(x) u v \, dx \quad \forall v \in H^1$$

$$F_f(v) = \int_\Omega f v \, dx + \int_{\partial\Omega} \alpha g v \, ds.$$

Assume $a_1 \geq a(x) \geq a_0 > 0$.

$$|a(u,v)| \leq |\alpha| \{ \| \nabla u \|_{L^2} \| \nabla v \|_{L^2}$$

$+$

$$a(u,u) = \int_\Omega \left(\alpha |\nabla u|^2 + a(x) |u|^2 \right) dx$$

$$\geq \int_\Omega \alpha |\nabla u|^2 + a_0 |u|^2 \, dx$$

$$\geq \min\{\alpha, a_0\} \int_\Omega |\nabla u|^2 + |u|^2 dx$$

$$= c \| u \|_{H^1}^2 .$$

$a(\cdot,\cdot)$ bdd in $H^1 \times H^1$.

$|F_f(v)|$

$$\leq \| f \|_{L^2} \| v \|_{L^2} + \alpha \| g \|_{L^2(\partial\Omega)} \| v \|_{L^2(\partial\Omega)}$$

$$\left(\overset{\text{Trace}}{\gamma : H^1(\Omega) \longrightarrow L^2(\partial\Omega)} \quad \| u \|_{L^2(\partial\Omega)} \leq C(n,p) \| u \|_{H^1(\Omega)} \right)$$

$|F_f(v)|$

$$\leq \| f \|_{L^2(\Omega)} \| v \|_{L^2(\Omega)} + \alpha C(n,p) \| g \|_{L^2(\partial\Omega)} \| v \|_{H^1}.$$

$$\leq \left[\| f \|_{L^2} + C(\alpha, n, 2) \| g \|_{L^2(\partial\Omega)} \right] \| v \|_{H^1}$$

$\Rightarrow \|F_f\|_{(H^1)^*} \leq K.$

By LLM $\exists!\, u \in H^1$

$\ni a(u, v) = (f, v)$

$\forall v \in H^1.$

20/12/19.

$$\partial_\rho L(t,\rho) = \frac{1}{2}\Big[M_g(\rho - ct) + (\rho - ct)\partial_\rho M_g(\rho - ct) + M_g(\rho+ct) + (\rho+ct)\partial_\rho M_g(\rho+ct)\Big] + \frac{1}{2c}\Big[(\rho+ct)M_h(\rho+ct) - (\rho-ct)M_h(\rho-ct)\Big]$$

$M(t,\rho) + \rho\,\partial_\rho M(t,\rho) =$ ''

$$= \frac{1}{2}\Big[\frac{1}{4\pi}\int_{S(0,1)} g(x+(\rho-ct)z)\,dw + (\rho-ct)\int_{S(0,1)}\frac{1}{4\pi}\nabla g(x+(\rho-ct)\cdot z\,dw + \cdots\Big]$$

$$+ \frac{1}{2c}\Big[\frac{(\rho+ct)}{4\pi}\int_{S(0,1)} h(x+(\rho+ct)z)\,dw - \frac{(\rho-ct)}{4\pi}\int_{S(0,1)} h(x+(\rho-ct)z)\,dw$$

let $\rho \longrightarrow 0$

$$u(t,x) = \frac{1}{8\pi}\int_{S(0,1)} g(x-ct)\,dw(z) + \frac{1}{8\pi}\int_{S(0,1)} g(x+ct\,z)\,dw(z)$$

$$+ \frac{-ct}{8\pi}\int_{S(0,1)} \nabla g(x-ct)\cdot z\,dw + \frac{ct}{8\pi}\int_{S(0,1)} \nabla g(x+ct)\cdot z\,dw$$

$$+ \frac{1}{8\pi c}\Big[ct\int_{S(0,1)} h(x+ctz)\,dw(z) + ct\int_{S(0,1)} h(x-ctz)\,dw(z)$$

$$= \frac{1}{4\pi}\int_{S(0,1)} g(x+ctz)\,dw(z) + \frac{ct}{4\pi}\int_{S(0,1)} \nabla g(x+ctz)\cdot z\,dw(z) + \frac{t}{4\pi}\int_{S(0,1)} h(x+ctz)\,dw$$

$$\partial_t\Big(\frac{t}{4\pi}\int_{S(0,1)} g(x+ctz)\,dw\Big) = \frac{1}{4\pi}\int_{S(0,1)} g(x+ctz)\,dw(z) + \frac{t}{4\pi}\int_{S(0,1)} \nabla g(x+ctz)\cdot cz\,dw.$$

$$u(x,t) = \partial_t\Big(\frac{t}{4\pi}\int_{S(0,1)} g(x+ctz)\,dw\Big) + \frac{t}{4\pi}\int_{S(0,1)} h(x+ctz)\,dw.$$

let $y = x + ctz$.

$$u(t,x) = \partial_t\left(\frac{t}{4\pi}\int_{S(x,ct)} g(y)\,\frac{d\sigma}{c^2t^2}\right)$$

$$+\frac{t}{4\pi}\int_{S(x,ct)} h(y)\,\frac{d\sigma}{c^2t^2}.$$

$$= \partial_t\left(\frac{1}{4\pi c^2 t}\int g(y)\,d\sigma\right)$$

$$+\frac{1}{4\pi c^2 t}\int_{S(x,ct)} h(y)\,d\sigma.$$

Kirchoff's formula.

Domain of dependence of (x,t) is $S(x,ct)$

* Hadamard's method of descent.

$$\partial_t^2 u - \Delta u = 0 \quad \text{in } (0,\infty)\times\mathbb{R}^2$$

$$u = g \quad \text{on } \{0\}\times\mathbb{R}^2.$$

$$\partial_t u = h \quad \text{on } \{0\}\times\mathbb{R}^2.$$

$$u = u(t, x_1, x_2)$$

See the problem as a 3D problem which is independent of the 3rd variable.

ie, $u = \tilde{u}(t, x_1, x_2, x_3)$

$$\partial_t^2\tilde{u} - c^2\Delta\tilde{u} = 0 \quad \text{in } (0,\infty)\times\mathbb{R}^3$$

$$\tilde{u} = \tilde{g} \quad \text{on } \{0\}\times\mathbb{R}^3$$

$$\partial_t\tilde{u} = \tilde{h} \quad \text{on } \{0\}\times\mathbb{R}^3$$

Let $x = (x_1, x_2) \in \mathbb{R}^2$

$$S = \{(y_1,y_2,y_3) \ni \sqrt{(y_1-x_1)^2+(y_2-x_2)^2+y_3^2} = ct\}$$

Sphere $\to ((x_1,x_2,0), ct)$
↓
radius.

$$u(t,x) = \tilde{u}(t,x_1,x_2,0)$$

$$= \partial_t\left[\frac{1}{4\pi c^2 t}\int_S g(y)\,d\sigma\right]$$

$$+\frac{1}{4\pi c^2 t}\int_S h(y)\,d\sigma$$

Let $S^+ = \{y\in S;\ y_3>0\}$
(Projecting the sphere to plane)

On S^+

$$y_3 = \sqrt{c^2t^2-(y_1-x_1)^2-(y_2-x_2)^2}$$

where $(y_1,y_2)\in D(x,ct)$

Disk centred at $x = (x_1,x_2)$ and radius ct

$$\int_S \tilde{g}(y)\,d\sigma = 2\int_{S^+}\tilde{g}(y)\,d\sigma$$

$$= 2\int_{S^+} g(y)\,d\sigma$$

Let $y_3 = \phi(y_1,y_2)$

$$= \sqrt{c^2t^2-(y_1-x_1)^2-(y_2-x_2)^2}$$

be a parametrisation of S^+.

$$d\sigma = \sqrt{1+\nabla\phi^2}\,dy_1\,dy_2$$

$$= \frac{ct}{\sqrt{c^2t^2-(y_1-x_1)^2-(y_2-x_2)^2}}\,dy_1\,dy_2$$

$$u(x,t) = \partial_t \left(\frac{2}{4\pi c^2 t} \int_{D(x,ct)} g(y_1,y_2) \frac{ct \quad dy_1 dy_2}{\sqrt{c^2t^2-(y_1-x_1)^2-(y_2-x_2)^2}} \right)$$

$$+ \frac{2}{4\pi c^2 t} \int_{D(x,ct)} h(y_1,y_2) \frac{ct \quad dy_1 dy_2}{\sqrt{c^2t^2-(y_1-x_1)^2-(y_2-x_2)^2}}$$

$$= \partial_t \left(\frac{1}{2\pi c} \int_{D(x,ct)} \frac{g(y_1,y_2) \quad dy_1 dy_2}{\sqrt{c^2t^2-(y_1-x_1)^2-(y_2-x_2)^2}} \right)$$

$$+ \frac{1}{2\pi c} \int_{D(x,ct)} \frac{h(y_1,y_2) \quad dy_1 dy_2}{\sqrt{c^2t^2-(y_1-x_1)^2-(y_2-x_2)^2}}$$

Domain of dependence is $D(x,ct)$

Well posedness of IVPs

$$\partial_t^2 u - c^2 \partial_x^2 u = 0 \text{ in } (0,\infty)\times\mathbb{R} \quad -(1)$$

$$u = g \quad \text{on } \{0\}\times\mathbb{R} \quad -(2)$$

$$\partial_t u = h \quad \text{on } \{0\}\times\mathbb{R} \quad -(3)$$

where $g \in C^2(\mathbb{R})$ and $h \in C^1(\mathbb{R})$ is given

To find a f^n $u \in C^2((0,\infty)\times\mathbb{R}) \cap C([0,\infty)\times\mathbb{R})$ satisfying (1), (2) & (3)

continuous dependence:

Given an $\varepsilon > 0$ $\exists$ $\delta > 0$ $\ni$ $\|g-\tilde{g}\| < \delta$ and $\|h-\tilde{h}\| < \delta$

$\Rightarrow \|u-\tilde{u}\| < \varepsilon$

D'Alembert's formula gives a soln which continuously depends on the data.

$$-\Delta u = f \quad \text{on } \Omega$$

$$u = 0 \quad \text{on } \partial\Omega.$$

$u \in H_0^1(\Omega)$, from weak formulatn

$$\Rightarrow |u|^2_{H^1(\Omega)} = (f,u)$$

$$\leq \|f\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)}$$

Poincare's

$$\Rightarrow c\|u\|^2_{L^2(\Omega)} \leq \|f\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)}$$

$$\Rightarrow \|u\|_{H^1(\Omega)} \leq c\|f\|_{L^2(\Omega)}$$

$(0,T)\times\mathbb{R}$

Ex show that

$$E(t) = \int_{\mathbb{R}} \left[(\partial_t u)^2 + c^2(\partial_x^2 u)^2\right] dx$$

is constant.

$E'(t) = 0$.

(Showing the existence ~~by~~ explicitly and energy method can be used to show the uniqueness)

SGH

Finding soln

$\frac{\partial u}{\partial t} - \Delta u = 0$ / $u_{tt} - \Delta u = 0$

1) Semigroup theory
2) Galerkin approximation
3) Time splitting algorithm

$$Lu = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + \sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i} + Cu$$

(Divergence & non-divergence form)

Self adjoint operator.

$A : H \longrightarrow H$.

$\langle Ax, y\rangle = \langle x, Ay\rangle$

$T \in B(H_1, H_2)$

$N(T) = R(T^*)^{\perp}$

$N(T^*) = R(T)^{\perp}$

$\overline{R(T)} = N(T^*)^{\perp}$

$\overline{R(T^*)} = N(T)^{\perp}$

1) $T^* = T$ self adj.
2) $\|Tx\| = \|x\|$ isometry.
3) $T^2 = T$ projection.

Def.

let H is separable Hilbert space

$\{\phi_1, \phi_2, \dots\}$ be onb

then $T = [a_{ij}]$

$a_{ij} = \langle T\phi_i, \phi_j\rangle$

let H_1, H_2 be infinite dim.

for $u \in H_1$, $v \in H_2$

$T : H_1 \longrightarrow H_2$

$T(x) = \langle u, x\rangle v$

Range$(T) = $ span$\{v\}$

||ly $u_1, u_2, \dots, u_n \in H_1$

$\{\phi_1, \phi_2, \dots, \phi_n\} \in H_2$

$T(x) = \sum_{i=1}^{n} \langle x, u_i\rangle \phi_i \ \forall x \in H_1$

$T \in B(H_1, \cancel{H_2})$

T is finite rank if $R(T)$ is f.dim.

<u>Theorem</u>

let $K : H_1 \longrightarrow H_2$ be a bdd linear operators of rank 'n'. Then $\exists$ $u_1, u_2, \dots, u_n \in H_1$ & $\phi_1, \phi_2, \dots, \phi_n \in H_2$ for every $x \in H_1$

$$K(x) = \sum_{i=1}^{n} \langle x, u_i\rangle \phi_i$$

$\phi_1, \phi_2, \cdots, \phi_n$ may be choosen as an onb of H_2.

Proof
let $\phi_1, \phi_2, \cdots, \phi_n$ be a basis of $R(T)$

$$Kx = \sum_{i=1}^{n} \langle x, \phi_i \rangle \phi_i$$

$f_i(x) = \langle Kx, \phi_i \rangle \Rightarrow f_i \in H_1^*$.

f_i is bdd linear. By RRT

$\Rightarrow \exists u_i \in H_1 \ni$ $\langle x, u_i \rangle = \langle Kx, \phi_i \rangle$

$$f_i(x) = \langle x, u_i \rangle.$$

$$\Rightarrow K(x) = \sum_{i=1}^{n} \langle x, u_i \rangle \phi_i.$$

Compact operator.

eg: 1) $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$

2) every finite rank operator is compact.

Arzela – Ascoli theorem is true for $C(X, \mathbb{R}^n)$.

3) Isometry is compact iff it has finite rank.

4) Identity map is compact if dim of this space is finite.

let $A : H_1 \to H_2$ is compact operator $B : H_2 \longrightarrow H_3$ is bdd then $\overset{AB}{BA}$ is compact.

Thm $T \in B(H_1, H_2)$

1) T is compact iff $TT^* = T^*T$ is compact.

2) T is compact iff T^* is compact.

▶ The set of finite rank operator is dense in compact operators.

Def $\lambda \in \mathbb{C}$ is called eigenvalue of $T \in B(H)$ if $\exists x \neq 0 \ni$

$$Tx = \lambda x.$$

▶ let $T \in B(H)$ be self adj

1) eigenvalues are real.

2) eigenvalues corresponding to distinct eigenvalues are orthogonal.

$$\lambda \langle x, y \rangle = \langle \lambda x, y \rangle = \langle Tx, y \rangle$$
$$= \langle x, Ty \rangle = \langle x, \mu y \rangle = \mu \langle x, y \rangle$$

3) $T \in B(H)$ is self adj

$\langle Tx, x \rangle = 0 \quad \forall x \in H$

$\Rightarrow T \equiv 0$.

Thm

let $T \in B(H)$ be self adj

1) let $\lambda = \inf\limits_{\|x\|=1} \langle Tx, x\rangle$,

If $\exists x_0 \in H \ni \|x_0\| = 1$ and $\lambda = \langle Tx_0, x_0\rangle$ then λ is an eigen value of T.

2) let $\lambda = \sup\limits_{\|x\|=1} \langle Tx, x\rangle$, If $\exists x_1 \in H \ni \|x_1\| = 1$ and $\lambda = \langle Tx_1, x_1\rangle$ then λ is an eigen value.

let $t \in \mathbb{C}$, $v \in H$ we've by def of λ

$$\langle T(x_0+tv), x_0+tv\rangle \geq \lambda \langle x_0+tv, x_0+tv\rangle$$

$$\langle Tx_0, x_0\rangle + 2\,\mathrm{Re}\, t \langle Tx_0, v\rangle + |t|^2 \langle Tv, v\rangle$$
$$\geq \lambda \langle x_0, x_0\rangle + 2\lambda\, \mathrm{Re}\, t \langle x_0, v\rangle + \lambda |t|^2 \langle v, v\rangle$$

let $\lambda = \langle Tx_0, x_0\rangle$ &

$t = \gamma \langle v, (T-\lambda I)x_0\rangle$ $\gamma \in \mathbb{R}$

$\Rightarrow \langle v, (T-\lambda I)x_0\rangle = 0 \quad \forall v \in H$

$\Rightarrow Tx_0 = \lambda x_0$.

If $T \in B(H)$ is compact self adj then atleast one of the $-\|T\|$ or $\|T\|$ is eigen values.

Proof

By earlier result if T is comp $\exists \{x_n\} \subset H$ with $\|x_n\| = 1$ for each n.

$\Rightarrow \lambda \in \mathbb{R}$ $\langle Tx_n, x_n\rangle \to \lambda$

where $\lambda = -\|T\|$ or $\|T\|$

Now, $\|Tx_n - \lambda x_n\|^2 = \langle\, ,\, \rangle$

$$= \|Tx_n\|^2 + \lambda^2 - 2\lambda \langle Tx_n, x_n\rangle$$
$$\leq 2\lambda^2 - 2\lambda \langle Tx_n, x_n\rangle$$
$$\to 0. \qquad \to 0.$$

$\Rightarrow Tx_{n_i} - \lambda x_{n_i} \to 0$

$x_n \to \frac{1}{\lambda} y$.

$y = \lim\limits_{n\to\infty} Tx_{n_i} = \frac{1}{\lambda} Ty$

$\Rightarrow Ty = \lambda y$.

Spectral Thm for S.A.C.O.

Suppose $T \in K(H)$ be self adj. Then $\exists$ a system of orthonormal eigen vectors $\{\phi_1, \phi_2, \ldots\}$ of T and

the corresponding eigen values $\lambda_1, \lambda_2 \ldots$
$\ni |\lambda_1| \geq |\lambda_2| \geq \ldots$

$$Tx = \sum_K \lambda_K \langle x, \phi_K \rangle \phi_K \quad \forall x \in H.$$

If $\{\lambda_K\}$ is infinite set then
$\lambda_K \to 0$ as $n \to \infty$.

Proof
T is self-adj compact
$T_1 = T$ of $H_1 = H$.
By previous result $\exists$ eignvalue $\lambda_1 \ni$
let $|\lambda_1| = \|T_1\|$ and
eigenvector $\ni \|\phi_1\| = 1$

$$\Rightarrow T_1\phi_1 = \|T_1\| \phi_1$$

$\because$ H is Hilbert

$$H_1 = \text{span}\{\phi_1\} \oplus \text{span}\{\phi_1\}^{\perp}$$

$H_2 = \text{span}\{\phi_1\}^{\perp}$ is closed
subset of H

$$\Rightarrow T_2 = T\big|_{H_2} \qquad T(H_2) \subset H_2.$$

$T_2 : H_2 \to H_2$ compact self adj.

Applying the above thus

$$|\lambda_2| = \|T_2\| \leq \|T_1\| = |\lambda_1|$$

$$\Rightarrow |\lambda_2| \leq |\lambda_1|$$

$$H = \text{span}\{\phi_1, \phi_2\} \oplus \text{span}\{\phi_1, \phi_2\}^{\perp}$$

$$H_3 = \text{span}\{\phi_1, \phi_2\}^{\perp}.$$

$T_3 : H_3 \to H_3$ is self adj compact

$$\Rightarrow \exists |\lambda_3| = \|T_3\| \leq \|T_2\|$$

$$T_3 \phi_3 = \lambda_3 \phi_3 \quad \|\phi_3\| = 1.$$

If $\{\lambda_n\}$ is infinite & $\lambda_n \not\to 0$
then $|\lambda_n| > \varepsilon$

If $n \neq m$

$$\|T\phi_n - T\phi_m\|^2 = \|\lambda_n \phi_n - \lambda_m \phi_m\|$$
$$> |\lambda_n| + |\lambda_m| > 2\varepsilon$$

$\to \leftarrow$ T is compact.

$$\Rightarrow \lambda_n \to 0.$$

Case i $\quad T_n = 0$ for some n.

let $\quad x_n = x - \sum_{K=1}^{n} \langle x, \phi_K \rangle \phi_K$

$$T(x_n) = 0 \Rightarrow Tx - \sum_{K=1}^{n} \lambda_K \langle x, \phi_K \rangle \phi_K = 0$$

Case ii
$T_n \neq 0$ for infinitely many n

$$\left\| Tx - \sum_{K=1}^{n} \lambda_K \langle x, \phi_K \rangle \phi_K \right\|$$
$$= \|T_n x_n\| = |\lambda_n| \|x_n\| \to 0.$$

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Observat's

1) $N(T)^{\perp} = \text{span}\{\phi_1, \phi_2, \dots, \phi_n \dots\}$

2) $H = N(T) \oplus N(T)^{\perp}$

$= N(T) \oplus \overline{R(T)}$.

3) If $H = N(T) \oplus_{K=1}^{\infty} G_K$.

$G_{\cancel{k}} = H_i^{\perp} = \oplus_{i=1}^{m} N(T_j - \lambda_i I)$

4) λ_i repeats only finitely many times.

multiplicity of $\lambda_i = p_i = \dim(N(T - \lambda_i I))$

Now let Lu, L is the elliptic operator defined earlier.

let $b_i = 0$ & $c = 0$.

$\Rightarrow L$ is self adj

$Lu = f$ in Ω

$u = 0$ in $\partial\Omega$

$f \in L^2(\Omega) \Rightarrow \exists!\ u \in H_0^1(\Omega)$.

let $K : L^2(\Omega) \xrightarrow[\text{bdd}]{} H_0^1(\Omega) \xrightarrow[\text{compact}]{} L^2(\Omega)$

$u = K(f)$.

$H_0^1(\Omega) \xhookrightarrow{i} L^2(\Omega)$

compact embedding.

$K : L^2 \to L^2$ is compact.

$LK(f) = f$ $\quad u \in H_0^1(\Omega)$

$a(u, v) = \int f v$.

$a(K(f), v) = \int f v$. $\forall v \in H_0^1(\Omega)$

let $g \in L^2(\Omega)$

$v = K(g)$

$a(K(f), K(g)) = \int f K(g)$

$a(K(g), K(f)) = \int g K(f)$

$\Rightarrow \int g K(f) = \int f K(g)$

$\Rightarrow K$ is ~~self~~ symmetric

If $K(f) = 0$

$a(K(f), v) = \int f v$ then

$0 = \int f v$.

$\Rightarrow f = 0$.

$\Rightarrow \ker K = \{0\}$.

$\Rightarrow \text{Range}(K) = L^2$.

$L\phi = \lambda\phi \Leftrightarrow \phi = L^{-1}\lambda\phi$

$\Rightarrow \phi = \lambda L^{-1}\phi$ ($\because LK(f) = f$)

$= \lambda K\phi$

$\Rightarrow K\phi = \frac{1}{\lambda}\phi$. $\lambda \neq 0$

say $K\phi = \mu\phi$

$\because K$ is C.S.A.O

by spectral theorem

$\exists\ |\mu_1| \geq |\mu_2| \geq \dots$ &

$\{\phi_1, \phi_2 \dots\} \in L^2(\Omega)$

$\Rightarrow K(\phi_i) = \mu_i \phi_i \ni \langle \phi_i, \phi_j \rangle = \delta_{ij}$
$- ①$

$L^2(\Omega) = \oplus \text{ span}\{\phi_i\}$

$f \in L^2$
$\Rightarrow f = \sum_K \langle f, \phi_i \rangle \phi_i$

by ①

$LK\phi_i = \phi_i$

$\Rightarrow L\mu_i \phi_i = \phi_i$

$\Rightarrow \mu_i L\phi_i = \phi_i$

$L\phi_i = \frac{1}{\mu_i} \phi_i$

$L\phi_i = \lambda_i \phi_i$

$L\phi_i = \lambda_i \phi_i \qquad a(u,v) = \int fv.$

$a(\phi_i, \phi_j) = \int \lambda_i \phi_i \phi_j$

$a(\phi_i, \phi_j) = \lambda_i \delta_{ij}$

$\Rightarrow \phi_i$ are orthogonal basis in $H_0^1(\Omega)$.

NB.

Hadamard
- 1902
- Well posedness

$\log y = 2\log x\, c.$
$y = x^2 c$ -
$y(0) = 1$
$1 = 0.$
$\Rightarrow$.

$C[a,b] \quad , \quad C'[a,b]$

$f_1(x) = f(x) + A \sin nx$

? $u'' = f(x)$
$u(0) = 0$
$u'(0) = 0$

? How we check stability of D'Alembert's eqn?

? $u'' = r(t) \quad , \quad u(0) = 0 = u'(0)$

$u_n(t) = u(t) + \ ? \qquad r_n(t).$

$r_n(t) = r(t) + \ ? \qquad 1/n$

eg: for wellposed and illposed problems.

$u_{tt} - c^2 u_{xx} = 0.$
$u(x,0) = f$
$u_t(x,0) = g.$

$|f(\cdot) - \tilde{f}(\cdot)| < \delta/2$
$|g(\cdot) - \tilde{g}(\cdot)| < \delta/2$

$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\,dy$

$\frac{1}{2}\left[f(x+ct) - \tilde{f}(x+ct) + f(x-ct) - \tilde{f}(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct}(g(y) - \tilde{g}(y))\,dy$

$\leq \frac{1}{2}\left(\frac{\delta}{2} + \frac{\delta}{2}\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \delta/2\, dy. \qquad \begin{matrix} x+ct \\ -x+ct \end{matrix}$

$= \delta + \frac{1}{2c}\,\delta/2\,(ct)$

$\Rightarrow K(\phi_i) = \mu_i \phi_i \ni \langle \phi_i, \phi_j \rangle = \delta_{ij}$

$- ①$

$L^2(\Omega) = \oplus \operatorname{span}\{\phi_i\}$

$f \in L^2$

$\Rightarrow f = \sum_K \langle f, \phi_i \rangle \phi_i$

by ①

$LK\phi_i = \phi_i$

$\Rightarrow L\mu_i \phi_i = \phi_i$

$\Rightarrow \mu_i L\phi_i = \phi_i$

$L\phi_i = \frac{1}{\mu_i} \phi_i$

$L\phi_i = \lambda_i \phi_i$.

$L\phi_i = \lambda_i \phi_i$ $\quad a(u, v) = \int f v.$

$a(\phi_i, \phi_j) = \int \lambda_i \phi_i \phi_j$

$a(\phi_i, \phi_j) = \lambda_i \delta_{ij}$

$\Rightarrow \phi_i$ are orthogonal basis in $H_0^1(\Omega)$.

NB.

Hadamard

- 1902

- Well posedness

$\log y = 2\log x c.$

$y = x^2 c$

$y(0) = 1$

$1 = 0.$

$C[a,b]$, $C'[a,b]$

$f_1(x) = f(x) + A \sin nx$

? $u'' = f(x)$

$u(0) = 0$

$u'(0) = 0$

? How we check stability of D'Alembert's eqn?

? $u'' = r(t)$, $u(0) = 0 = u'(0)$

$u_n(t) = u(t) + ?$ $\quad \frac{r(t)}{n}$

$r_n(t) = r(t) + ?$ $\quad \frac{1}{n}$

eg: for wellposed and illposed problems.

$u_{tt} - c^2 u_{xx} = 0.$

$u(x,0) = f$

$u_t(x,0) = g.$

$|f(\cdot) - \tilde{f}(\cdot)| < \delta/2$

$|g(\cdot) - \tilde{g}(\cdot)| < \delta/2$

$u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\,dy$

$\frac{1}{2}[f(x+ct) - \tilde{f}(x+ct) + f(x-ct) - \tilde{f}(x-ct)]$

$+ \frac{1}{2c}\int_{x-ct}^{x+ct} (g(y) - \tilde{g}(y))\,dy$

$\leq \frac{1}{2}\left(\frac{\delta}{2} + \frac{\delta}{2}\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \delta/2 \, dy.$ $\quad \begin{matrix} x+ct \\ -x+ct \end{matrix}$

$= \delta + \frac{1}{2c}\,\delta/2\,(ct)$

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~~Hault~~

Gronwall's inequality.

Diff.

$\eta'(t) \le \phi(t)\eta(t) + \psi(t)$,

$\eta(t) \le e^{\int_0^t \phi(s)ds}\left[\eta(0) + \int_0^t \psi(s)ds\right]$

$\eta'(t)\, e^{\phi(t)}$

$\eta'(t) - c_1\eta(t) \le c_2$.

$e^{-c_1 t}(\eta'(t) - c_1\eta(t)) \le c_2 e^{-c_1 t}$.

Integral

$\xi(t) \le c_1 \underbrace{\int_0^t \xi(s)ds}_{\eta(t) \le \eta(0)+} + c_2$.

$\eta'(t) = \xi(t)$.

$\xi(t)$

$L^p(0,T;X)$

$C(0,T;X)$

$W^{m,p}(0,T;X)$

Heat eqn. fundamental soln.

$u(x_1,t_1) \le c\, u(x_2,t_2) \quad 0<t_1<t_2$

Harnack's inequality.

~~5G1~~ Lax-Milgram.

$Lu = f$.

weak formulation of L

$B[u,v] = I_1 + I_2 + I_3 = \int f v$ OR

$(I_1+I_2+I_3) \ge I_1 - (|I_2|+|I_3|)$

$|B[u,u]| \ge \alpha \|u\|^2_{H^1_0} - \gamma\|u\|^2_{L^2}$

$B_\mu[u,v]$,

for continuity poincaré's

- $Lu + \lambda u = f$

Without using Lax-Milgram how we solve

$$\left.\begin{aligned} Lu &= f \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega \end{aligned}\right\}$$

→ Fredholm Alternative

$K: H \longrightarrow H$ is compact op.

Let $T = I - K$.

1) $N(T)$ is finite dim

2) $R(T)$ is closed

3) $N(T) = \{0\} \Leftrightarrow R(T) = H$

4) $\dim N(T) = \dim R(T^*)$

5) $N(T)^\perp = R(T^*)$

$(L_\lambda u = Lu + \lambda u = f \text{ in } \Omega)$

1) $Lu = f$ in Ω

$u = 0$ on $\partial\Omega$

(has soln if $L^* v = 0$ in Ω

$v = 0$ on $\partial\Omega$

iff $(f, v) = 0$

i) $Lu=0$, $u=0$ on $\partial\Omega$ has nonzero soln. iff $u-\tilde{K}u=0$ in Ω

$u=0$ on $\partial\Omega$

has zero soln.

$L_\lambda : H_0^1(\Omega) \longrightarrow L^2(\Omega)$

let $K_\lambda(f)=u$.

for given $f\in L^2(\Omega)$ $u\in H_0^1(\Omega)$

$K_\lambda(f): L^2(\Omega) \longrightarrow H_0^1(\Omega) \overset{i}{\hookrightarrow} L^2(\Omega)$

$K_\lambda : L^2 \longrightarrow L^2$

$L_\lambda u = f$.

$L_\lambda K_\lambda(f) = f$.

$K_\lambda(f) = L_\lambda^{-1} f$.

$Lu=f$.

$\Rightarrow Lu+\lambda u = f+\lambda u$.

$L_\lambda u = f+\lambda u$.

$u = L_\lambda^{-1}(f) + L_\lambda^{-1}(\lambda u)$

$u = K_\lambda(f) + \lambda L_\lambda^{-1}(u)$

$u+h = K_\lambda(f)$ f

$\tilde{K} = \lambda L_\lambda^{-1}$

$u = h + \tilde{K}v$.

$\Rightarrow u - \tilde{K}u = h$.

$u-\tilde{K}u = h$ in Ω

$u=0$ on $\partial\Omega$

if $(v,h)=0$ where $v-\tilde{K}^* v=0$)

$h = L_\lambda^{-1} f$.

$\Rightarrow K_\lambda f = h$

$(K_\lambda f, v) = 0$

$(f, K_\lambda^* v) = 0$.

$(f, \tilde{K}^* v) = 0$.

$\Rightarrow (f,v) = 0$. $\forall v$.

$H_0^1 \subset L^2 \subset H^{-1}$

$f\in H^{-1}$

$\Rightarrow f = \tilde{f} - \sum f_{x_i}$ $f_{x_i}\in L^2(\Omega)$

$Lu+\lambda I = f$.

$(L+\lambda I)$:

<u>Regularity</u>

L is the general elliptic operator.

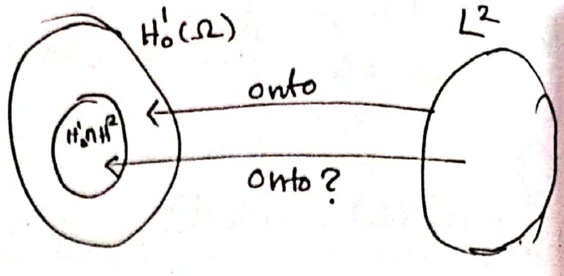
$Lu = f$ on Ω

$u=0$ on $\partial\Omega$.

$f\in L^2(\Omega)$ $\exists!$ $u\in H_0^1(\Omega)$.

Tomorrow

Is our soln u better than a member of $H^1_0(\Omega)$?



Assume $a_{ij}=1$ $b_i=0$ $c=0$.

$$\int \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} = \int f v \quad \forall v \in H^1_0$$

$$\Rightarrow -\int \frac{\partial^2 u}{\partial x_i \partial x_j} v = \int f v \quad \forall v \in H^1_0$$

$$\Rightarrow \int \left(f + \frac{\partial^2 u}{\partial x_i \partial x_j}\right) v = 0 \quad \forall v \in H^1_0(\Omega)$$

$-\Delta u = f$. $f \in L^2$

$$\int_{\mathbb{R}^n} f^2 = \int_{\mathbb{R}} |\Delta u|^2$$

$$= \int_{\mathbb{R}} u_{x_i x_i} u_{x_j x_j}$$

$$= -\int_{\mathbb{R}} u_{x_i x_i x_j} u_{x_j}$$

$$= \int_{\mathbb{R}} |u_{x_i x_j}|^2 dx$$

$$\int_{\mathbb{R}^n} f^2 = \int_{\mathbb{R}^n} |D^2 u|^2 dx.$$

$\Rightarrow u \in H^2(\Omega)$

$f \in C^1 \Rightarrow u \in C^3$

$f \in C^2 \Rightarrow u \in C^4$

$\vdots$

$f \in C^\infty \Rightarrow u \in C^\infty$.

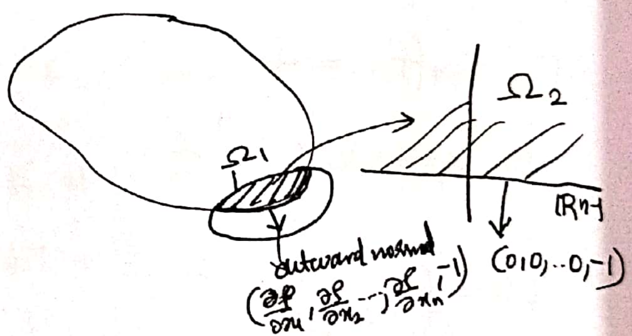
I) Interior regularity $\Omega = \mathbb{R}^n$

II) Boundary regularity

$\Omega = \mathbb{R}^n_+$

$\partial\Omega = \{x \mid x = (x_1, x_2, \dots, x_{n-1}, 0)\}$

Assume that $\partial\Omega$ is smooth



Boundary can be represented by $\rho : \mathbb{R}^{n-1} \longrightarrow \mathbb{R}$.

$\text{graph}(\rho) = \{(x', \rho(x'))\}$

$\Omega_1 = \{x \mid x_n > \rho(x')\}$

$F : \Omega \longrightarrow \mathbb{R}^n_+$

$F(x) = y$.

$y_1 = x_1$

$y_2 = x_2$

$\vdots$

$y_{n-1} = x_{n-1}$

$y_n = x_n - \rho(x')$

let $H^1(\Omega) = \{u \in L^2(\Omega) \,/\, \nabla u \in L^2(\Omega)\}$

If $u \in H^1(\Omega) \Rightarrow F(u) \in H^1(\Omega_2)$

$$\begin{array}{ccc} \Omega_1 & \xrightarrow{f} & \Omega_2 \\ & & \downarrow \\ & & \mathbb{R} \end{array}$$

$F_x(u) = u(F(x))$

let $u \in L^2(\Omega)$

$$\int_\Omega |F_x(u)|^2 dx = \int_\Omega |u(F(x))|^2 dx$$

Put $y = F(x)$

$dy = |\det F'| \, dx$.

$\Rightarrow dx = \dfrac{dy}{|\det F'|}$

$$\Rightarrow \int_\Omega |F_x(u)|^2 dx = \int_{\Omega_1} \frac{|u(y)|^2 dy}{|\det F'|}$$

$$= c\int_{\Omega_2} |u|^2 dy < \infty.$$

$F_x \in L^2$

$u \in H^m(\Omega_1)$

$\Rightarrow F_x(u) \in H^m(\Omega_2)$.

If $c_1 \le |\det F'| \le c_2$

then $F_x(u) \in L^2$

$\to u \in L^2$.

Difference quotient.

27/12/19

NB.

- Holder's inequality
- Minkowski inequality
- Cauchy - Schwartz.

$|(u,v)| \le \|u\| \|v\|$

- Gronwall's inequality

$\eta'(t) \le \phi(t)\eta(t) + \psi(t)$

$\eta \to$ absolutely continuous.

$\phi, \psi \to$ summable f^ns

$$\boxed{\eta(t) \le e^{\int_0^t \phi(s)ds}\left[\eta(0) + \int_0^t \psi(s)ds\right]}$$

$\xi(t)$ non-negative summable

$\xi(t) \le c_1 \int_0^t \xi(s)ds + c_2$

Then $\boxed{\xi(t) \le c_2\left[1 + c_1 t e^{c_1 t}\right]}$

let $\eta(t) = \int_0^t \xi(s)ds$

$\eta(0) = 0 \quad \eta'(t) = \xi(t)$.

$\Rightarrow \eta'(t) \le c_1\eta(t) + c_2$.

$$\eta(t) \le e^{\int_0^t c_1 ds}\left[\eta(0) + \int_0^t c_2 ds\right]$$

$$= e^{c_1 t} c_2 t.$$

$$\xi(t) \le c_1 \eta(t) + c_2$$

$$\le c_2\left[1 + c_1 t e^{c_1 t}\right].$$

- Interpolation inequality

Assume $1 \le s \le r \le t \le \infty$ and

$\frac{1}{r} = \frac{\theta}{s} + \frac{(1-\theta)}{t}$

$u \in L^s(U) \cap L^t(U) \Rightarrow u \in L^r(U)$

$\|u\|_{L^r} \le \|u\|_{L^s}^{\theta} \|u\|_{L^t}^{1-\theta}$.

$p = \frac{s}{\theta r}$, $q = \frac{t}{(1-\theta) r}$.

$\int |u|^r dx \quad \theta = s/pr \quad (\int |u|^s)^{\frac{s}{pr}}$

$\int uv \le (\int u^s)^{...}$

$(\int |u|^r dx)^{1/r} \le (\int |u|^s dx)^{\theta/s} (\int |u|^t)^{\frac{1-\theta}{t}}$

$1 = \frac{1}{p} + \frac{1}{q}$. $\quad p\theta r$. $\quad \frac{1}{p}+\frac{1}{q}$

$q(1-\theta) r = t$.

$\int |u|^{r + \theta r - \theta r} dx$.

$= \int |u|^{\theta r} \cdot |u|^{(1-\theta) r} dx$.

$\le (\int |u|^{\theta r p})^{1/p} (\int |u|^{(1-\theta) r q})^{1/q}$

$= (\int |u|^s)^{1/p} (\int |u|^t)^{1/q}$.

$= (\int |u|^s)^{\frac{\theta r}{s}} (\int |u|^t)^{\frac{(1-\theta) r}{t}}$

$= \|u\|_s^{\theta r} \|u\|_t^{(1-\theta) r}$

$\Rightarrow \|u\|_r \le \|u\|_s^{\theta} \|u\|_t^{(1-\theta)}$.

- $L^p(0,T;X)$

$L^\infty(0,T;X)$

$C(0,T;X)$

$W^{1,p}(0,T;X)$

$h : [0,T] \longrightarrow X$.

$U f(x,t) = h(t) \quad u(x,t)$.

$u(x)$

$\left(\int_0^t \|u\|_X^p \, dx \right)^{1/p} < \infty$.

ess sup $\|u\|_X$

max $\|u\|_X$.

$u_t + Lu = f \qquad U \times (0,T] = U_T$

$u(x,0) = g$

$u = 0 \qquad$ on ∂U_T

$Lu = -\sum_{i,j=1}^{n} (a_{ij}(x,t) u_{x_i})_{x_j}$

$+ \sum_{i=1}^{n} b_i(x,t) u_{x_i} + c(x,t) u$.

$a_{ij}, b_i, c \in L^\infty(U_T)$

$f \in L^2(U_T)$.

$u_t + \Delta u = f$.

$(u_t, v) + (\Delta u, v) = (f, v)$

$(u_t, v) - (\nabla u, \nabla v) = (f, v)$

$u \in H^1$

$(u_t, v) + (u, \Delta v) = (f, v)$

$u_t + Lu = f$ }

$(u_t, v) + (Lu, v) = (f, v)$

$(u_t, v) + B[u, v; t] = (f, v)$

$\forall v \in H_0^1(\Omega)$

$\downarrow$

to pass derivative

$v \in C^1([0,T]; H_0^1(\Omega))$

denote $u_t = u'$.

$(u', v) + B[u, v; t] = (f, v)$

SG

$Lu = f$

$L \to$ general elliptic operator.

Let $a_{ij} \in C^1(\Omega)$

$b_j, c \in L^\infty$

$f \in L^2(\Omega)$

$\Rightarrow \exists!\ u \in H_0^1(\Omega)$

$\Rightarrow u \in H^2(\Omega)$

▸ Let $u \in L^2(\Omega)$ and we want to S.T.

$\nabla u \in L^2(\Omega)$

It is enough to S.T

$$\tau_t(u(x)) = \frac{u(x+te_j) - u(x)}{t}$$

& $\|\tau_t u(x)\| \le C$, const.

$\tau_t \to$ Difference quotient.

Lemma

Let $u \in H^1(\mathbb{R}^n)$ and $\{e_j\} \in \mathbb{R}^n$

$$v_t(x) = \frac{u(x+te_j) - u(x)}{t}$$

$$\to \frac{\partial u}{\partial x_j} \text{ as } t \to 0.$$

Proof

Let $u \in C_c^\infty(\mathbb{R}^n)$

$$u(x+te_j) - u(x) = \int_0^1 \frac{d}{d\theta} u(x+\theta t e_j)\, d\theta$$

$$= \int_0^1 \frac{\partial u}{\partial x_j} u(x+\theta t e_j)\, t\, d\theta$$

$$\frac{u(x+te_j) - u(x)}{t} = \int_0^1 \frac{\partial u(x+\theta t e_j)}{\partial x_j}\, d\theta$$

take modulus and square on both the sides, then integrating

$$\int_{\mathbb{R}^n} \left| \frac{u(x+te_j) - u(x)}{t} \right|^2 dx$$

$$\le \int_{\mathbb{R}^n} \int_0^1 \left| \frac{\partial}{\partial x_j} u(x+\theta t e_j) \right|^2 d\theta\, dx.$$

Put $y = x + \theta t e_j$

$$\leq \int_{\mathbb{R}^n} \int_0^1 \left|\frac{\partial u}{\partial y_i}(y)\right|^2 d\theta\, dy$$

$$\leq \int_{\mathbb{R}^n} \left|\frac{\partial u}{\partial y_i}(y)\right|^2 dy.$$

$$\Rightarrow \int_{\mathbb{R}^n} \left|\frac{u(x+te_j)-u(x)}{t}\right|^2 dx$$

$$\leq \int_{\mathbb{R}^n} |\nabla u|^2 dx$$

$$\Rightarrow \|v_t\|_{L^2(\mathbb{R}^n)} \leq \|\nabla u\|_{L^2(\mathbb{R}^n)}$$

$\forall u \in C_c^\infty(\mathbb{R}^n)$

$\because$ $C_c^\infty(\mathbb{R}^n)$ is dense in $H^1(\mathbb{R}^n)$

$$\Rightarrow \|v_t\|_{L^2(\mathbb{R}^n)} \leq \|\nabla u\|_{L^2(\mathbb{R}^n)}$$

$$\|v_t\|_{L^2(\mathbb{R}^n_+)} \leq \|\nabla u\|_{L^2(\mathbb{R}^n_+)}$$

for $1 \leq i \leq n-1$.

is also true.

Theorem

let L be the general elliptic operator

$a_{ij} \in C^1(\mathbb{R}^n)$, $b_i, c \in L^\infty(\mathbb{R}^n)$

$f \in L^2(\mathbb{R}^n)$, $u \in H^1(\mathbb{R}^n)$

$\ni$ $Lu = f$ in the weak sense

then,

$u \in H^2(\mathbb{R}^n)$,

Further $\exists\, c > 0 \ni$

$$\|u\|_{H^2(\mathbb{R}^n)} \leq c\|f\|_{L^2} + \|u\|_{L^2}$$

Aim $\|\tau_t \nabla u\| \leq c.$

Claim $\frac{u(x+te_j)-u(x)}{t} \to \frac{\partial u}{\partial x_j}$ in D'

let $\phi \in C_c^\infty(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} \left(\frac{u(x+te_j)-u(x)}{t}\right) \phi(x)\,dx.$$

$$= \frac{1}{t} \int_{\mathbb{R}^n} \left(u(x+te_j)\phi(x) - u(x)\phi(x)\right) dx$$

~~let $y = x$~~

$$= \frac{1}{t}\left[\int_{\mathbb{R}^n} u(y)\,\phi(y-te_j)\,dy - \int_{\mathbb{R}^n} u(y)\phi(y)\,dy\right]$$

$$\int \phi\, \frac{u(x+te_j)-u(x)}{t}\,dx$$

$$= -\int_{\mathbb{R}^n} u(y)\left[\frac{\phi(y)-\phi(y-te_j)}{t}\right] dy$$

$\because u \in H^1(\mathbb{R}^n) \Rightarrow u \in L^2$

$\Rightarrow$ by DCT

$$\lim_{t\to 0} \int_{\mathbb{R}^n} \left[\frac{u(x+te_j)-u(x)}{t} \right] \phi(x)\, dx$$

$$= -\int_{\mathbb{R}^n} u(x) \frac{\partial \phi}{\partial x_j}\, dx \quad -(1)$$

let $v_t = \dfrac{u(x+te_j)-u(x)}{t}$

wkt $\|v_t\|_{L^2} \leq \|\nabla u\|_{L^2} \leq C$

$\because u \in H^1(\mathbb{R}^n)$

By Banach Alagoulu thm

$v_t \longrightarrow v$ in $L^2(\mathbb{R}^n)$

$$\lim_{t\to 0} \int v_t \psi\, dx = \int v\psi$$

$$\psi \in L^2(\mathbb{R}^n) \quad -(2)$$

from (1) & (2)

$$\int_{\mathbb{R}^n} v\phi\, dx = -\int_{\mathbb{R}^n} u \frac{\partial \phi}{\partial x_j}\, dx$$

v is a weak derivative of u.

$u \in H^1_0(\Omega)$

$$\tilde{u} = \begin{cases} u, & x \in \Omega \\ 0, & x \notin \Omega \end{cases}$$

$\tilde{u} \in H^1(\mathbb{R}^n)$

Observatns

$K = \operatorname{supp}(u)$ compact $\subset \mathbb{R}^n$

let $\tau_t(u) = \dfrac{u(x+te_j)-u(x)}{t}$

$u \in H^1(\mathbb{R}^n) \Rightarrow \tau_t(u) \in H^1(\mathbb{R}^n)$

- D.Q is invariant under translation.

$$\frac{\partial}{\partial x} \tau_t(u) = \tau_t \frac{\partial u}{\partial x}$$

$$\frac{\partial}{\partial x_j} \tau_t(u) = \frac{\partial}{\partial x_j} \left[\frac{u(x+te_j)-u(x)}{t} \right]$$

$$= \frac{\partial}{\partial x_j}(x+te_j) - \frac{\partial u(x)}{\partial x_j}$$

$$= \tau_t \frac{\partial u}{\partial x}.$$

$$\sum a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} + \sum b_i \frac{\partial u}{\partial x_i} v + cuv\, dx$$

$$= \int f v \quad v \in H^1(\mathbb{R}^n)$$

$$\Rightarrow \sum_{i,j} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} = \int f_1 v\, dx \quad -(3)$$

$\forall v \in H^1(\mathbb{R}^n)$

$$f_1 = f - \sum b_i \frac{\partial u}{\partial x_i} - cu.$$

$f_1 \in L^2(\mathbb{R}^n)$

$\because u \in H^1(\mathbb{R}^n)$ & $b_i \in L^\infty(\mathbb{R}^n)$

$f \in L^2(\mathbb{R}^n)$

let $v = \tau_{-t} \tau_t(u)$

take v as above in (3).

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$$\sum_{i,j}\int a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial(\tau_{-t}\tau_t u)}{\partial x_j}dx$$

$$= \int f_1\, \tau_{-t}\tau_t u\, dx.$$

$$\text{RHS} = \int f_1\, \tau_{-t}\tau_t u\, dx$$

$$\le \left(\int_{\mathbb{R}^n}|f_1|^2\right)^{1/2}\left(\int_{\mathbb{R}^n}|\tau_{-t}\tau_t u|^2 dx\right)^{1/2}$$

let $\omega = \tau_t u$.

$$\le \left(\int_{\mathbb{R}^n}|f_1|^2\right)^{1/2}\left(\int_{\mathbb{R}^n}|\tau_{-t}\omega|^2 dx\right)^{1/2}$$

$$= \|f_1\|_{L^2}\,\|\tau_t\omega\|_{L^2}$$

$$\le \|f_1\|_{L^2}\,\|\nabla\omega\|_{L^2}$$

$$\text{RHS} \le \|f_1\|_{L^2}\,\|\nabla\tau_t u\|_{L^2}$$

$$(\because \|\tau_t u\|_{L^2} \le \|\nabla u\|_{L^2})$$

$$\text{LHS} = \sum_{ij}\int_{\mathbb{R}^n} a_{ij}\frac{\partial u}{\partial x_j}\,\tau_t\frac{\partial}{\partial x_j}(\tau_t u)\,dx$$

Wkt, $\int \tau_t\phi\psi\, dx = \int \phi\, \tau_t\psi\, dx$

$$= \sum\int_{\mathbb{R}^n}\tau_t\left(a_{ij}\frac{\partial u}{\partial x_j}\right)\left(\tau_t\frac{\partial u}{\partial x_j}\right)dx$$

$$\tau_t(\phi\psi) = \frac{\phi(x+te_i)\psi(x+te_i)-\phi(x)\psi(x)}{t}$$

$$= \phi(x+te_i)\,\tau_t\psi + \psi(x)\,\tau_t\phi(x)$$

$$= \sum_{i,j}\int_{\mathbb{R}^n}\Big[a_{ij}(x+te_i)\,\tau_t\frac{\partial u}{\partial x_j} + \frac{\partial u}{\partial x_j}\tau_t(a_{ij}(x))\Big]\tau_t u_{x_j}\,dx$$

$$= I_1 + I_2.$$

$$I_1 = \sum_{i,j}\int_{\mathbb{R}^n} a_{ij}(x+te_j)\,\tau_t\frac{\partial u}{\partial x_i}\,\tau_t\frac{\partial u}{\partial x_j}$$

$$\ge m\|\tau_t\nabla u\|_{L^2}^2 \quad \text{— (1).}$$

$$|I_2| = \left|\sum_{i,j}\int_{\mathbb{R}^n}\frac{\partial u}{\partial x_j}\,\tau_t a_{ij}(x)\,\tau_t\frac{\partial u}{\partial x_j}\,dx\right|$$

$$\le c\left(\int_{\mathbb{R}^n}\left|\frac{\partial u}{\partial x_i}\right|^2\right)^{1/2}\left(\int_{\mathbb{R}^n}\left|\tau_t\frac{\partial u}{\partial x_i}\right|^2\right)^{1/2}$$

$$\le c\left(\int|\nabla u|^2\right)^{1/2}\left(\int(\nabla\tau_t u)^2\right)^{1/2}$$

$$\text{LHS} = I_1 - |I_2|$$

$$\text{LHS} = M\|\tau_t\nabla u\|_{L^2}^2 - C\|\nabla u\|_{L^2}^2\|\nabla\tau_t u\|_{L^2}$$

LHS = RHS

$$\Rightarrow \|f_1\|_{L^2}\|\nabla\tau_t u\|_{L^2}$$

$$\ge m\|\tau_t\nabla u\|_{L^2}^2$$

$$- C\|\nabla u\|_{L^2}^2\|\tau_t\nabla u\|_{L^2}^2$$

$$\Rightarrow \|\tau_t\nabla u\|_{L^2} \le c\Big(\underbrace{\|f_1\|_{L^2}+\|\nabla u\|_{L^2}}_{\text{bdd.}}\Big)$$

$$\because f_1, \nabla u \in L^2$$

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∇u is bdd.

$\Rightarrow$ ∇u converges weakly

$\because x_n \rightharpoonup x \Rightarrow \|x\| \le \liminf \|x_n\|$

$\|\frac{\partial}{\partial x_i} \nabla u\| \le \liminf_{t\to 0} \|\nabla \tau_t u\|_{L^2}$

$\le c(\|f\|_{L^2} + \|\nabla u\|_{L^2})$

$\|\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} u\|_{L^2} \le c(\|f\|_{L^2} + \|\nabla u\|_{L^2})$

$\Rightarrow u \in H^2(\mathbb{R}^n)$.

- $a_{ij} \in C^2$, $b_i, c \in C^1$, $f \in H^1$

$\Rightarrow u \in H^3$

- Now consider $\Omega \subset \mathbb{R}^n$ bdd open

$Lu = f$, $f \in L^2$.

$\Rightarrow u \in H_0^1(\Omega)$

choose $\chi \in C_c^\infty(\mathbb{R}^n)$

let $v = \chi u \in H_0^1(\Omega)$

$$\frac{\partial v}{\partial x_i} = \frac{\partial}{\partial x_i}\chi u + \chi \frac{\partial u}{\partial x_i}$$

$$\frac{\partial^2 v}{\partial x_i \partial x_j} = \frac{\partial^2 \chi}{\partial x_i \partial x_j} u + \frac{\partial \chi}{\partial x_i}\frac{\partial u}{\partial x_j} + \frac{\partial u}{\partial x_j}\frac{\partial \chi}{\partial x_j} + \chi \frac{\partial^2 u}{\partial x_i \partial x_j}$$

$Lu = f \Rightarrow$

$$-\sum a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_i \left(b_i - \frac{\partial a_{ij}}{\partial x_j}\right)\frac{\partial u}{\partial x_i} + cu = f.$$

$\beta = \frac{\partial^2 \chi}{\partial x_i \partial x_j}$

$\alpha = \frac{\partial \chi}{\partial x_j} + \frac{\partial \chi}{\partial x_j}$

previous eqn will become

$$\Rightarrow \chi f_1 + \sum \alpha_i \frac{\partial u}{\partial x_i} + \beta u = -\sum_{ij} a_{ij}\frac{\partial^2 v}{\partial x_i \partial x_j}$$

$\Rightarrow v \in H_0^1(\Omega)$.

let U be open $\ni \bar{U}$ is compact & $\bar{U} \supset\supset U$

$\Rightarrow v = u \in H^2(U)$ choosing $\chi = 1$ on U

$\Rightarrow u \in H^2_{loc}(\Omega)$

NB.

$u_t + Lu = f(x,t)$ in U_T.

$u(x,0) = g(x)$ in U

$u \in L^2(0,T; H_0^1(U))$

$u' \in L^2(0,T; H^{-1}(U))$

i) $(u', v) + B[u, v; t] = (f, v)$

ii) $u(0) = g$.

$f \in H^{-1}(U)$, then

$\exists f^0, f^1, \dots, f^n \in L^2(U)$.

$$\langle f, v \rangle = \int f^0 v \, dx + \int \sum_{i=1}^n f^i v_{x_i} dx$$

$$\|f\|_{H^{-1}} = \inf \left\{ \left(\sum (f^i)^2\right)^{1/2} \Big/ f^i \in L^2 \right\}$$

$$u_t = f - Lu .$$

$$= \underbrace{f - \sum b^i u_{x_i} - cu + \sum (a^{ij} u_{x_i})_{x_j}}_{g^0 + \sum_{j=1}^n g^i_{x_j}}$$

~~$(u_t, v) =$~~

i) Construction of approximate solution.

ii) Derivation of energy estimates for approximate soln

iii) convergence of approximate soln.

$\{w_k\}$ onb of L^2. $L^2(\Omega)$

$$u_m(t) = \sum_{k=1}^m d_m^k(t) w_k(x).$$

$$d_m^k(0) = (g, w_k)$$

$$(u_m', v) + B[u_m, v, t] = (f, v)$$

Let $v = w_k$.

$$(u_m', w_k) + B[u_m, w_k, t] = (f, w_k)$$

$$d_m^{k'}(t) = (f, w_k) - B[u_m, w_k, t]$$

Thm

For each integer $m = 1, 2, \dots$

$\exists$ a unique f^n u_m of the form (1), satisfying (2) & (3).

$$d_m^{k'}(t) = (f, w_k) - B[u_m, w_k, t]$$

$$B[u_m, w_k, t]$$

$$= B\left[\sum_{l=1}^m d_m^l(t) w_l, \dots w_k ; t\right]$$

$$= \sum_{l=1}^m d_m^l(t) \underbrace{B[w_l, \dots w_k ; t]}_{\text{denote } e^{kl}}$$

$$= \sum_{l=1}^m e^{kl} d_m^l(t)$$

$$d_m^{k'}(t) + \sum_{l=1}^m e^{kl} d_m^l(t) = f_k .$$

$$d_m^k(0) = (g, w_k)$$

Thm

$$u_m \in L^\infty(0, T ; L^2(U))$$

$$u_m \in L^2(0, T ; H_0^1(U))$$

$$u_m' \in L^2(0, T ; H^{-1}(U))$$

$$\max_{0 \le t \le T} \|u_m\|_{L^2(U)} + \|u_m\|_{L^2(0,T; H_0^1(U))}$$

$$+ \|u_m'\|_{L^2(0,T; H^{-1}(U))}$$

$$\le C \left[\|f\|_{L^2(0,T; L^2(U))} + \|g\|_{L^2(U)} \right]$$

$(u_m', w_K) + B[u_m, w_K; t] = (f, w_K)$ (1) $\Rightarrow$

$(u_m', u_m) + B[u_m, u_m; t] = (f, u_m)$

$(u_m', u_m) = \frac{1}{2}\frac{d}{dt} \|u_m\|^2.$

Wkt,

$\beta \|u_m\|^2_{H_0^1(U)} \leq B[u_m, u_m; t] + \gamma \|u_m\|^2_{L^2(U)}$

$(f, u_m) \leq \frac{1}{2}\|f\|^2_{L^2} + \frac{1}{2}\|u_m\|^2_{L^2}$

$\Rightarrow \frac{1}{2}\frac{d}{dt}\|u_m\|^2 + \beta\|u_m\|^2_{H_0^1(U)}$

$\leq \frac{1}{2}\|f\|^2_{L^2(U)} + \left(\frac{1}{2}+\gamma\right)\|u_m\|^2_{L^2}.$ —— (1)

$\frac{d}{dt}\|u_m\|^2_{L^2} \leq 2\,\text{LHS}.$

$\Rightarrow \frac{d}{dt}\|u_m\|^2_{L^2} \leq \|f\|^2_{L^2} + (1+2\gamma)\|u_m\|^2_{L^2}$

$\eta(t) = \|u_m\|^2_{L^2}.$

$\eta(0) = \|u_m(0)\|^2_{L^2}$ $\left[\begin{array}{l} u_m(0) = \sum d_m^K(0) w_K \\ = \sum \langle g, w_K \rangle w_K \end{array}\right.$

$= \|u_m(0)\|^2_{L^2}$

$= \|g\|^2_{L^2}$

Gronwall's inequality

$\Rightarrow \eta(t) \leq C(T)\left[\|g\|^2_{L^2} + \|f\|^2_{L^2([0,T]; L^2(U))}\right]$

$\|u_m\|^2_{L^2} \leq C(T)\left[\|g\|^2_{L^2} + \|f\|^2_{L^2([0,T]; L^2(U))}\right]$

$\Rightarrow$ take sup on both sides
$\sup_t$

$u_m \in L^\infty(0,T; L^2(U)).$

$2\beta \|u_m\|^2_{H_0^1(U)} + \frac{d}{dt}\|u_m\|^2$

$\leq \frac{1}{2}\|f\|^2_{L^2(U)} + \|u_m\|^2_{L^2}$

$+ 2\gamma \|u_m\|^2_{L^2}.$

$\leq$ ~~$\frac{1}{2}\|f\|^2_{L^2} + \|u_m\|^2_{L^2}$~~

~~$+ 2\gamma\|\nabla u_m\|^2_{L^2}$~~

$\leq C\left[\|f\|^2_{L^2(U)} + \|g\|^2_{L^2} + \|f\|^2_{L^2(0,T; L^2(U))}\right]$

integrating we get

$\|u_m(T)\|^2_{L^2} + 2\beta\int_0^T \|u_m\|^2_{H_0^1(U)} dt$

$\leq C(T)\left[\|f\|^2_{L^2(0,T; L^2(U))} + \|g\|^2_{L^2}\right]$

$\Rightarrow u_m \in L^2(0,T; H_0^1(U)).$

choose

$v \in H_0^1$, $v = w + z.$

$w \in \text{span}\{w_1, \dots, w_m\} = V_m$

$z \in V_m^\perp.$

replace w_K by w.

$(u_m', w) + B[u_m, w; t]$

$= (f, w)$

$\langle u_m', v \rangle$ $v \in H_0^1$

$\Rightarrow u_m' \in H^{-1}$

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$\Rightarrow u_m \longrightarrow u$ in $L^2[0,T; H_0^1(U)]$

$u_m' \longrightarrow u'$ in $L^2[0,T; H^{-1}(U)]$

$u_m(0) = g$

$\Rightarrow u(0) = g.$

Consider

$(u_m', v) + B[$

SG

Uniform ellipticity cannot be relaxed in solving elliptic eqns.

Unif. ellipticity condition and graph of soln.

Hopf lemma.

Harnack inequality.

→ Keller_ossermann

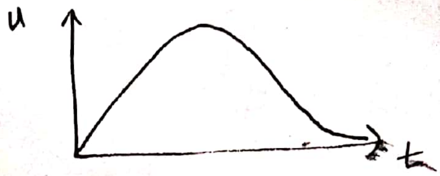
→ De-Georgi-Nash-Moser (~~AMS~~ AMS review)

28/12/19

LGG.

Diffusion

Heat profile



Pendulum, fundamental solns

Bridge

solns of basic eqns.

D'Alembert's

→ completely depending on initial data.

(1) — $u_{tt} - c^2 \Delta u = f$ in $Q_T = \Omega \times (0,T)$

$u(x,0) = 0$ on $\Sigma_T = \partial\Omega \times (0,T)$

$u(x,0) = g(x)$

$u_t(x,0) = h(x)$ in $\Omega \times (t=0)$

unknown

$u : \bar{\Omega} \times (0,T) \longrightarrow \mathbb{R}$

known

$f : Q_T \longrightarrow \mathbb{R}$

$g : \bar{\Omega} \longrightarrow \mathbb{R}$

$h : \bar{\Omega} \longrightarrow \mathbb{R}.$

1D — Wave eqn

2D — membrane problem

3D — elastic solid

Weak formulation $\quad -\int_{\partial\Omega} v \frac{\partial u}{\partial \nu} ds$

~~Met~~ let $v \in C_0^\infty(\Omega)$

$$\int_\Omega u_{tt} v \, dx + c^2 \int_\Omega \nabla u \nabla v \, dx$$

$$= \int_\Omega f v \, dx.$$

$u \in L^2(0,T; H^1_0(\Omega))$
$u_t \in L^2(0,T; L^2(\Omega))$
$u_{tt} \in L^2(0,T; H^{-1}(\Omega))$ $\Bigg\}$ (2)

$u \in H^2(0,T; X)$

$\|u\|^2_{H^2(0,T;X)} = \int_0^T \|u\|^2_X + \|u_t\|^2_X + \|u_{tt}\|^2_X$

(2) $\Rightarrow$ $u \in C([0,T]; L^2(\Omega))$

$u_t \in C([0,T]; H^{-1}(\Omega))$

We want,

$f \in L^2(Q_T)$

$v \in H^1_0(\Omega)$

Def.

Given $f \in L^2(Q_T)$, $g \in$

$h \in$

to find $u \in L^2(0,T; H^1_0(\Omega))$

$\ni u_t \in L^2(0,T; L^2(\Omega))$,

$u_{tt} \in L^2(0,T; H^{-1}(\Omega))$

satisfying,

i) $\langle u_{tt}, v\rangle + c^2 \langle \nabla u, \nabla v\rangle$

$= (f, v)$ $\quad v \in H^1_0(\Omega)$

ii) $u(0) = g$, $u_t(0) = h$.

For every $v \in H^1_0(\Omega)$

$w(t) = \langle u_{tt}, v\rangle$ in $\mathcal{D}'(0,T)$

$w(t) = \frac{d^2}{dt^2}\langle u, v\rangle$ in $\mathcal{D}'(0,T)$

For every $\psi \in \mathcal{D}(0,T)$

$\int_0^T w(t)\psi(t)dt = \int_0^T \langle u, v\rangle \ddot{\psi}(t)dt$

Thm

A f^n $f : [0,T] \to H$ is summable iff $t \longmapsto \|f\|_H$ is summable on $[0,T]$

$(u, \int_0^T f(t)dt)_H = \int_0^T (u, f(t))_H$

$u \in H$.

$u \in H^*$

$\int_0^T w(t)\psi(t)dt$

$= \int_0^T \langle u_{tt}, v\rangle \psi(t)dt$

$= \langle \int_0^T u_{tt}\psi, v\rangle$

$= \langle \int_0^T \cancel{u_{tt}}\, u\, \psi_{tt}, v\rangle$

$= \int_0^T (u, v)\psi_{tt}\, dt$.

$\int_0^T \frac{d^2}{dt^2}(u(t), v)\psi\, dt = $ "

(1) can be rewritten as,

$$\frac{d^2}{dt^2}(u,v) + c^2(\nabla u, \nabla v) = (f,v)$$
$$\forall v \in H_0^1(\Omega)$$

Motivation for Faedo-Galerkin Method

$$\left.\begin{array}{lll} u_t - \Delta u = 0 & \text{in} & \Omega \times (0,\infty) \\ u = 0 & \text{on} & \partial\Omega \times (0,\infty) \\ u = g & \text{in} & \Omega \times \{t=0\} \end{array}\right\} \text{(HE)}$$

let $u(x,t) = v(t)\,w(x)$

then the eqn become,

$$\frac{v'(t)}{v(t)} = \frac{\Delta w(x)}{w(x)} = \mu.$$

$\rightarrow v'(t) = \mu v(t)$,

$\Rightarrow v = c e^{\mu t}$

$\rightarrow \Delta w = \mu w$

two eigenvalue problems

Suppose λ is an eigenvalue

$\exists\ w \neq 0 \ni$

$$-\Delta w = \lambda w \text{ in } \Omega$$
$$w = 0 \text{ on } \partial\Omega.$$

then w is an eigenfunction.

$$u(x,t) = c e^{-\lambda t} w(x).$$

Solves (HE).

If $\{\lambda_k\}_{k=1,2,\dots m}$.

then by superposition principle

$$u_m = \sum_{k=1}^{m} c_k e^{-\lambda_k t} w_k.$$

is also a soln.

$$u_m(x,0) = \sum_{k=1}^{m} c_k w_k = g.$$

NB

$u_m \in L^\infty(0,T; L^2(U))$

$u_m \in L^2(0,T; H_0^1(U))$

$u_m' \in L^2(0,T; H^{-1}(U))$

$u_m \longrightarrow u \quad L^2(0,T; L^2(U))$

$u_m' \longrightarrow u' \quad \text{in } L^2(0,T; H^{-1}(U))$

$$(u_m', v) + B[u_m, v; t] = (f,v)$$

~~u(0)=g~~. ~~∀ v∈H~~

$\forall v \in L^2(0,T; H_0^1(U))$

$u(0) = g \qquad v \in C^1([0,T]; H_0^1(\Omega))$

let $v_N = \sum_{k=1}^{N} d_k(t) w_k.$

$$\int_0^T \langle u_m', v\rangle + B[u_m, v; t] = \int_0^T (f,v)$$

$$-\int_0^T \langle u_m, v'\rangle + B[u_m, v; t] = \int_0^T (f,v) - \langle u_m(0), v\rangle$$

To find energy estimate

$$\int_0^T (u_1, v') + B[u_1, v; t] = \int_0^T (f, v) - (u(0), v)$$

$\rightarrow u(0) = g$.

let u_1, u_2 be a solns

$\therefore$

$$-\int_0^T (u_1, v') + B[u_1, v; t] = \int_0^T (f, v) - (g, v)$$

$$\int_0^T ((u_1 - u_2)', v) + B[u_1 - u_2, v; t] = 0.$$

let $u = u_1 - u_2$

$\therefore$

$$(u', v) + B[u, v; t] = 0.$$

$v = u$.

$$(u', u) + B[u, u; t] = 0.$$

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + \beta \|u\|_{H_0^1} - \gamma \|u\|_{L^2} \le 0.$$

$$\Rightarrow \frac{d}{dt}\|u\|^2 \le \gamma \|u\|_{L^2}$$

By Gronwall's inequality

$$\eta(t) = \|u\|^2 = 0$$

$\Rightarrow u = 0 \rightarrow \underline{u_1 = u_2}$.

$u_t - \Delta u = f \quad \mathbb{R}^n \times (0, \infty)$

$u(x, 0) = g(x)$

$u(x, t) = 0$ on bdry

$$\int_{\mathbb{R}^n} f^2 dx = \int_{\mathbb{R}^n} (u_t - \Delta u)^2 dx.$$

$$= \int_{\mathbb{R}^n} u_t^2 - 2u_t \Delta u + (\Delta u)^2 dx.$$

$$= \int_{\mathbb{R}^n} u_t^2 + 2\int_{\mathbb{R}^n}$$

$$\int_{\mathbb{R}^n} (\Delta u)^2 dx = \sum \int_{\mathbb{R}^n} u_{x_i x_i} u_{x_j x_j} dx$$

$$= \sum \int_{\mathbb{R}^n} u_{x_i x_2}^2 dx$$

$$= \int_{\mathbb{R}^n} (D^2 u)^2 dx.$$

$$\sum \int u_t u_{x_i x_i} dx = \sum \int u_{t x_i} u_{x_i} dx.$$

$$= \int Du_t \, Du \, dx.$$

$$= \frac{1}{2}\frac{d}{dt} \int |Du|^2 dx.$$

$$\frac{d}{dt}\int |Du|^2 dx + \int_{\mathbb{R}^n} (D^2 u)^2 + \int_{\mathbb{R}^2} u_t^2 dx$$

$$= \int_{\mathbb{R}^n} f^2 dx.$$

$$\frac{d}{dt}\int |Du|^2 dx \le \int_{\mathbb{R}^n} f^2 dx.$$

Gronwall's inequality $\Rightarrow$ $u(0) = g$.

$$\int |Du|^2 dx \le \int |Dg|^2 + \int_0^T \int_{\mathbb{R}^n} f^2 dx dt$$

$$\max_{0 \le t \le T} \int |Du|^2 dx \le C(\;\; \text{"} \;\;)$$

$$\|Du\|_{C([0,T];L^2(\mathbb{R}^n))} \le c\left[\|Dg\|^2_{L^2(\mathbb{R}^n)} + \|f\|^2_{L^2(0,T;L^2(\Omega))}\right]$$

$$\int_0^T\int_{\mathbb{R}^n} |D^2u|^2\,dx\,dt + \int_0^T\int_{\mathbb{R}^n} u_t^2\,dx\,dt$$

$$+ \int_{\mathbb{R}^n} (Du)^2\,dx \le \int_0^T\int_{\mathbb{R}^n} |f|^2\,dx\,dt + \int |Dg|^2\,dx.$$

$$u_{tt} - \Delta u_t = f_t.$$

$$u_t(x,0) = -$$

$$u_t(x,t) = 0 \text{ on bdry.}$$

$$|u(x,t)| \to 0 \quad |x| \to \infty.$$

let $v = u_t$.

$$\left.\begin{aligned} &v_t - \Delta v = f_t \\ &v(x,0) = f(x,0) + \Delta g \\ &v(x,t) = 0 \quad \text{on } \cancel{\partial\Omega} \times (0,t) \end{aligned}\right\}$$

$$\int_{\mathbb{R}^n} v_t v - \int_{\mathbb{R}^n} \Delta v\cdot v = \int_{\mathbb{R}^n} f_t v$$

$$\Rightarrow \int_{\mathbb{R}^n} v_t v + \int_{\mathbb{R}^n} |Dv|^2\,dx = \int_{\mathbb{R}^n} f_t v.$$

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n} |v|^2 + \int_{\mathbb{R}^n} |\nabla v|^2\,dx = \int_{\mathbb{R}^n} f_t v\,dx$$

$$\le \int_{\mathbb{R}^n} \frac{f_t^2}{2}\,dx + \int_{\mathbb{R}^n} \frac{|v|^2}{2}\,dx \quad (*)$$

$$\frac{d}{dt}\int_{\mathbb{R}^n} |v|^2 \le \int_{\mathbb{R}^n} f_t^2\,dx + \int_{\mathbb{R}^n} |v|^2\,dx$$

Gronwall's

$$\Rightarrow \int_{\mathbb{R}^n} |v|^2\,dx \le e^t\left[\|v(0)\|_2^2 + \int_0^T\int_{\mathbb{R}^n} f_t^2\,dx\,dt\right]$$

$$V(0) = f(\cdot,0) + \Delta g.$$

~~integra~~

$$\int_{\mathbb{R}^n} |v|^2\,dx \le C(T)\left[\|f(\cdot,0)\|_2^2 + \|\Delta g\|_2^2 + \int_0^T\int_{\mathbb{R}^n} f_t^2\,dx\,dt\right]$$

Integrate $(*)$ over $(0,T)$

$$\int_{\mathbb{R}^n} |v|^2\,dx + 2\int_0^T\int_{\mathbb{R}^n} |Dv|^2\,dx\,dt$$

$$\le c\left[\int_{\mathbb{R}^n} |f_t|^2\,dx + \|f(\cdot,0)\|_2^2 + \|\Delta g\|_2^2 + \|f_t\|^2_{L^2(0,T;\mathbb{R}^n)}\right]$$

$\Rightarrow g \in H^2$

$\to$ Compatibility condition.

☹ Here to prove $u \in H^2$ we need to assume $u \in H^3$.

LSG

Eigenvalues of symmetric elliptic operator.

~~u_t~~ $- \cancel{\Delta u = f \text{ in } \Omega_T}$

Suppose

$$Lu = -\sum_{i,j=1}^n (a_{ij}u_{x_i})_{x_j}$$

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$a_{ij} = a_{ji}$

What's the advantage?

$a_{ij}(x) \in C^{\infty}(\Omega)$

Theorem

1) Each eigenvalue of L is real.

2) $\Sigma = \{\lambda_K\}_{K=1}^{\infty}$ where

$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots$

$\lambda_K \to \infty$ as $K \to \infty$.

3) $\exists$ orthonormal basis $\{w_K\}_{K=1}^{\infty}$ in $L^2(\Omega)$, where $\{w_K\} \in H_0^1(\Omega)$ is an eigenfn corresponding to λ_K

$$L w_K = \lambda w_K \quad \text{in } \Omega$$
$$w_K = 0 \quad \text{on } \partial\Omega.$$

$$u_m = \sum_{K=1}^{m} d_K(t)\, w_K$$

$d_K(t)$ to be determined.

Faedo - Galerkin.

$$u_m(t) = \sum_{K=1}^{m} d_m^K(t)\, w_K$$

where $d_m^K(0) = (g, w_K)$

$d_m^{K'}(0) = (h, w_K)$ $\quad K = 1, 2, \dots m.$

Find $u_m \in H^2(0,T; H_0^1(\Omega))$ such that for all,

$$(u_m''(t), w_K) + c^2(\nabla u_m, \nabla w_K) = (f, w_K)$$

$$u_m(0) = (g, w_K)$$
$$u_m'(0) = (h, w_K)$$

[Denote (f, w_K) as f^K]

Theorem

$\forall\, m \geq 1$, $\exists$ a unique solution to the wave equation.

Proof

$$(u_m''(t), w_K) = \left(\sum_{K=1}^{m} d_K''(t)\, w_K, w_K\right)$$
$$= d_K''(t) \quad [\because \{w_K\} \text{ is orthonormal}$$

$$c^2(\nabla u_m, \nabla w_K)$$
$$= c^2\left(\sum_{K=1}^{m} d_K(t)\, \nabla w_K, \nabla w_K\right)$$
$$= c^2(\nabla w_K, \nabla w_K)\, d_K(t)$$
$$= c^2 \|\nabla w_K\|_{L^2(\Omega)}^2\, d_K(t)$$

$$(f, w_K) = f^K, \quad K = 1, 2, \dots, m.$$

Now weak formulation become

$$d_K''(t) = -c^2 \|\nabla w_K\|_{L^2}^2\, d_K(t) + f^K \quad K = 1, 2, \dots, m$$

$d_K(0) = g$

$d_K'(0) = h.$

? $\exists$ $f^K \in L^2(0,T)$.

Since $f^K \in L^2[0,T]$ $K=1,2,\dots m$

$\exists$ a unique soln to ODE

$\therefore$ $d_K(t) \in H^2(0,T;\mathbb{R}^m)$

From the def,

$$u_m(t) = \sum_{K=1}^{m} d_K(t)\omega_K \in H^2(0,T;H_0^1(\Omega))$$

To prove

We are looking to give a bound

u_m in $L^\infty(0,T;H_0^1(\Omega))$

f in $L^2(0,T;H_0^1(\Omega))$

u_m' in $L^\infty(0,T;L^2(\Omega))$

u_m'' in $L^2(0,T;H^{-1}(\Omega))$

Thm

Let u_m be the soln of wave eqn. Then

$$\max_{0\le t\le T}\left\{\|u_m'\|^2_{L^2(\Omega)} + c^2\|u_m\|^2_{H_0^1(\Omega)}\right\}$$

$$\le c\left(\|h\|^2_{L^2(\Omega)} + \|\nabla g\|^2_{L^2(\Omega)} + \int_0^t \|f\|^2\,ds\right)$$

Substitute $u_m'(t)$ as test fn in weak formulatn.

$$(u_m'', u_m') + c^2(\nabla u_m, \nabla u_m') = (f, u_m')$$

for a.e. $t \in [0,T]$

$$(u_m'', u_m') = \frac{1}{2}\frac{d}{dt}\|u_m'\|^2_{L^2(\Omega)}$$

$$\left[\frac{d}{dt}(u_m, u_m) = (u_m', u_m) + (u_m, u_m')\right]$$

$$c^2(\nabla u_m, \nabla u_m')$$

$$= \frac{c^2}{2}\frac{d}{dt}\|\nabla u_m\|^2_{L^2(\Omega)}$$

$$(f, u_m') \le \|f\|_{L^2(\Omega)}\|u_m'\|_{L^2(\Omega)}$$

$$\le \frac{1}{2}\left(\|f\|^2_{L^2(\Omega)} + \|u_m'\|^2_{L^2(\Omega)}\right)$$

$$\frac{1}{2}\frac{d}{dt}\|u_m'\|^2_{L^2(\Omega)} + \frac{c^2}{2}\frac{d}{dt}\|\nabla u_m\|^2_{L^2(\Omega)}$$

$$\le \frac{1}{2}\left(\|f\|^2_{L^2} + \|u_m'\|^2_{L^2}\right)$$

? $\left[\int_s^t \frac{d}{dt}\|u(r)\|^2 dr = \|u(t)\|^2_{L^2} - \|u(s)\|^2_{L^2}\right]$

Integrating over $(0,t)$

$$\|u_m'\|^2_{L^2(\Omega)} + c^2\|\nabla u_m\|^2_{L^2(\Omega)}$$

$$\le \|u_m'(0)\|^2_{L^2} + c^2\|\nabla u_m(0)\|^2_{L^2} + \int_0^t\left(\|f\|^2_{L^2(\Omega)} + \|u_m'\|^2_{L^2(\Omega)}\right)ds$$

$\left((u_m'(0), u_m'(0)) = \|h\|_{L^2}^2\right)$

$\leq \|h\|_{L^2}^2 + c^2\|\nabla g\|_{L^2}^2$

$+ \int_0^t \|f\|_{L^2}^2 + \|u_m'\|_{L^2}^2 \, ds.$

$\leq C\left(\|h\|_{L^2}^2 + \|\nabla g\|_{L^2}^2 + \int_0^t \|f\|_{L^2}^2 + \int_0^t \|u_m'\|_{L^2}^2\right)$

Gronwall's inequality

$\eta(t) \leq G(t) + C\int_0^t \eta(s)\,ds$

then $\eta(t) \leq G(t)e^{ct}$.

let $\eta(t) = \|u_m'\|_{L^2(\Omega)}^2 + c^2\|\nabla u_m\|_{L^2(\Omega)}^2$

$G(t) = \|h\|_{L^2(\Omega)}^2 + \|\nabla g\|_{L^2(\Omega)}^2 + \int_0^t \|f\|^2 ds$

$\|u_m'\|_{L^2(\Omega)}^2 + c^2\|\nabla u_m\|_{L^2(\Omega)}^2$

$\leq ce^t\left(\|h\|_{L^2(\Omega)}^2 + \|\nabla g\|_{L^2(\Omega)}^2\right.$

$\left.+ \int_0^t \|f\|^2 ds\right)$

Taking max & poincaré's inequality

$u_m \in L^\infty(0,T; H_0^1(\Omega))$

$u_m' \in L^\infty(0,T; L^2(\Omega))$

SG1.

$\Delta u = f.$

Maximum principle holds.

consider the general elliptic problem operator,

Assume ellipticity $\exists\, \lambda, \Lambda > 0$

$\lambda|\nabla u|^2 \leq \sum a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} \leq \Lambda|\nabla u|^2$

$A = [a_{ij}]$. symmetric +ve definite

Hessian $[D^2u]$

trace $(AD^2u) = 0$

$\Leftrightarrow \quad \sum a_{ii}\frac{\partial^2 u}{\partial x_i \partial x_i} = 0.$

A is diagonalizable

$\exists\, OO^T = I \ni$

$OAO^T = D \quad d_i > 0.$

$\begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix} \begin{bmatrix} m_1 & & \\ & \ddots & \\ & & m_n \end{bmatrix} \quad m_i \in \mathbb{R}.$

let $m_1, \dots m_k > 0$ & $m_{k+1}, \dots, m_n \leq 0$

$Tr(A) = \sum_i \langle Ae_i, e_i\rangle$

$Tr(AD^2u) = \sum_i \langle AD^2e_i, e_i\rangle$

$= \sum_i \langle Am_ie_i, e_i\rangle$

$= \sum_i m_i\langle Ae_i, e_i\rangle$

$\lambda \leq a_{ij} \leq \Lambda.$

$m_i \lambda \le m_i a_{ij} \le \Lambda m_i$

$$\sum_{i=1}^{k} m_i \lambda \le \sum_{i=1}^{k} \langle Ae_i, e_i\rangle \le \sum_{i=1}^{k} \Lambda m_i . \quad m_i > 0 .$$

$$\sum_{i=k+1}^{n} m_i \lambda \le \sum_{i=k+1}^{n} \langle Ae_i, e_i\rangle \le \sum_{i=k+1}^{n} \Lambda m_i$$

$\Rightarrow tr(AD^2u) = (tr AD^2u)_+ + (tr AD^2u)_-$

$\Rightarrow \Lambda(D^2u)_- + \lambda(D^2u)_+ \le (tr AD^2u) \le \Lambda(D^2u)_+ + \lambda(D^2u)_-$

If $tr(AD^2u) = 0$ then

$\Lambda(D^2u)_- + \lambda(D^2u)_+ \le 0$ &

$\Lambda(D^2u)_+ + \lambda(D^2u)_- \ge 0$

$\Rightarrow \lambda \|D^2u\|_+ + \Lambda\|D^2u\|_- \le 0$

& $\Lambda\|D^2u\|_+ + \lambda\|D^2u\|_- \ge 0.$

$\|\cdot\| \to$ curvature

$\|D^2u\| = \sum m_i$

$\Rightarrow \lambda\|D^2u\|_+ \le \Lambda\|D^2u\|_-$, $Lu \le 0$

& $\Lambda\|D^2u\|_+ \le \lambda\|D^2u\|_-$ $Lu \ge 0$

Library book.

For the time being assume $b_i, c = 0$ in L.

$Lu \le 0$ concave.

$Lu \ge 0$ convex.

Weak max. principle

Let L the general elliptic operator with $c = 0$

Let Ω be a connected bdd openset in $\mathbb{R}^n$

$u \in C^2(\Omega) \cap C(\bar{\Omega})$,

If $Lu \le 0$ then $\max_{\Omega} u = \max_{\partial\Omega} u$

$Lu \ge 0$ then $\min_{\Omega} u = \min_{\partial\Omega} u$

Proof.

Let $Lu < 0$.

and $x_0 \in \Omega \ni u(x_0) = \max_{\Omega} u$.

$\Rightarrow Du(x_0) = 0$ at x_0.

& $D^2u(x_0) \le 0$

$\because$ L is elliptic & a_{ij} is symmetric

$\exists O \ni OAO^T = [d_1 \cdots d_n]$ $d_i > 0$

$y = x_0 + O(x - x_0)$

$\Rightarrow x - x_0 = O^T(y - x_0)$

$$u_{x_i} = \sum_{K=1}^{n} u_{y_K} O_{Ki}$$

$$\sum a_{ij} u_{x_i x_j} = \sum_{K=1}^{n} d_K\, u_{y_K y_K}$$

$c > 0$ in L

$\max_{\bar\Omega} u = \max_{\partial\Omega} u^+$

$\min_{\bar\Omega} u = \min_{\partial\Omega} u^-$

$u^+ = \max\{u, 0\}$

$u^- = \min\{u, 0\}$

$c < 0$ in $L \Rightarrow$ there is no maximum principle.

Hopf Lemma

~~strong~~ maximum ~~principle~~

Ω is connected bdd open domain, $u \in C^2(\Omega) \cap C(\bar\Omega)$

$c = 0$ in L.

If $Lu \le 0$ and $x_0 \in \partial\Omega$

and $u(x_0) > u(x)$ $\forall x \in \Omega$

then $\frac{\partial u}{\partial \nu} > 0$.

$\frac{\partial u}{\partial \nu} = \nabla u \cdot \hat{n} = \langle \nabla u, \frac{-x}{|x|} \rangle$

(If x_0 is a point of max. $Du(x_0) = 0$)

Ω has interior ball condition

Strong Maximum principle.

~~Hopf Lemma~~

Assume $u \in C^2(\Omega) \cap C^1(\bar\Omega)$

and $c \equiv 0$ in

$u_{m x_1} \quad u_{m x_2}$

$a^{ij} = a^{ji}$

Comparison principles (or maximum principles) are valid only for when ellipticity condition holds.!

Otherwise we use Harnack's inequality

Uniform ellipticity condit^n holds only if $[a^{ij}]$ is symmetric.

29/12/19.

NB

Improved regularity

$g(x) \in H_0^1(U)$ & $f \in L^2(0,T; L^2(U))$

$u \in L^2(0,T; H^2(U)) \cap L^\infty(0,T; H_0^1(U))$

$u' \in L^2(0,T; L^2(U))$

$$\sup \|u\|^2_{H_0^1(U)} + \int_0^T \|u\|^2_{H^2(U)} dt$$

$$+ \int_0^T \|u'\|^2_{L^2(U)} dt$$

$$\le C\left[\|f\|^2_{L^2(0,T; L^2(U))} + \|g\|^2_{H_0^1(U)} \right]$$

$$(u_m', u_m') + B[u_m, u_m'] = (f, u_m')$$

$B[u_m, u_m']$

$$= \int_U \underbrace{\sum_{i,j} a^{ij} u_{m x_i} u'_{m x_j}}_{A_1} + \underbrace{\sum_i b^i u_{m x_i} u_m' + c u_m u_m' \, dx}_{A_2}$$

$$A[u, v] = \int_U \sum a^{ij} u_{x_i} v_{x_j} \, dx.$$

$$A_1 = \int \frac{1}{2} \frac{d}{dt} \sum_{i,j} a^{ij} u_{m x_i} u_{m x_j} \, dx$$

$$= \frac{1}{2} \frac{d}{dt} [A[u_m, u_m]]$$

$A[u_m, u_m] \geq \theta \|u_m\|_{H_0^1(U)}$.

$|A_2| \leq \frac{1}{4}\|u_m'\|_{L^2}^2 + 4c\|u_{m x_i}\|_{L^2}^2$

$+ \frac{1}{4}\|u_m'\|_{L^2}^2 + 4c\|u_m\|_{L^2}^2$.

constant can be chosen accordingly.

$\|u_m'\|_{L^2(U)}^2 + \frac{d}{dt} A[u_m, u_m]$

$\leq c\left[\|f\|_{L^2}^2 + \|u_m\|_{H_0^1(U)}^2\right]$

$\int_0^T \|u_m'\|_{L^2}^2 + A[u_m, u_m]$

$\leq c\left[A[u_m(0), u_m(0)] + \|f\|_{L^2(0,T;L^2(U))}^2\right]$

$\int_0^T \|u_m'\|_{L^2}^2 dt + \theta\|u_m\|_{H_0^1(U)}^2$

$\leq c\left[\|g\|_{H_0^1(U)}^2 + \|f\|_{L^2(0,T;L^2(U))}^2 + \|u_m\|_{L^2(0,T;H_0^1(U))}^2\right]$

↓ Bound we've from existence.

$B[u,v] = (h,v)$ ~~$h = f - u'$~~.

let $h = f - u'$

$\|u\|_{H^2(U)}^2 \leq c(\|h\|_{L^2(U)}^2 + \|u\|_{L^2(U)}^2)$

$\leq c(\|f\|_{L^2}^2 + \|u'\|_{L^2}^2 + \|u\|_{L^2}^2)$

$\int_0^T \|u\|_{H^2(U)}^2 dt$

$\leq c(\|f\|_{L^2(0,T;L^2(U))}^2 + \|g\|_{H_0^1(U)}^2)$

$g(x) \in H^2(U)$, $f' \in C^2(0,T;L^2(U))$

$u \in L^\infty(0,T;H^2(U))$

$u' \in L^\infty(0,T;L^2(U)) \cap L^2(0,T;H_0^1(U))$

$u'' \in L^2(0,T;H^{-1}(U))$

$u_t + Lu = f$.

let $v = u_t$ and diff. the eqn wrt t,

$v_t + Lv = f_t$

$v(x,0) = (f - Lu)(x,0)$

$(v_t, w_k) + B[v, w_k; t] = (f_t, w_k)$

i.e., $(v_m', w_k) + B[v_m, w_k] = (f', w_k)$.

$B[v_m, v_m^*]$

$= \underbrace{\int \sum a^{ij} v_{m x_i} v_{m x_j} dx}_{A_1}$

$+ \underbrace{\int \sum b_i v_{m x_i} v_m^* dx + \int c v_m v_m^* dx}_{A_2}$.

~~$A_1 = \frac{1}{2}\frac{d}{dt}A[v_m, v_m]$~~.

$\beta\|u\|_{H_0^1}^2 \leq B[u,u] + \gamma\|u\|_{L^2}^2$.

$(v_m', v_m) + B[v_m, v_m] = (f', v_m)$

$\frac{d}{dt}\|v_m\|_{L^2}^2 + 2\beta\|v_m\|_{H_0^1(U)}^2$

$\leq \|f'\|_{L^2(U)}^2 + (1+2\gamma)\|v_m\|_{L^2(U)}^2$ —— (*)

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$$\|v_m\|_{L^2}^2 + 2\beta\int_0^T \|v_m\|^2_{H_0^1(U)}\,dt$$

$$\le c\left[\|v_m(0)\|_2^2 + \|f'\|^2_{L^2(0,T;L^2(\Omega))} + \|v_m\|^2_{L^2(0,T;L^2(U))}\right]$$

$$\frac{d}{dt}\|v_m\|_{L^2}^2 \le (1+2\gamma)\|v_m\|^2_{L^2(U)} + \|f'\|^2_{L^2(U)}.$$

$\eta(t) = \|v_m\|_{L^2}^2$.

$$\eta(t) \le e^{\int_0^t \phi(s)ds}\left[\eta(0) + \int_0^t \psi(s)ds\right]$$

$$\eta(t) \le e^{(1+2\gamma)T}\left[\|v_m(0)\|^2_{L^2} + \int_0^T \|f'\|^2_{L^2}\right]$$

$\Rightarrow$ (*) become,

$$\frac{d}{dt}\|v_m\|^2_{L^2} + 2\beta\|v_m\|^2_{H_0^1(U)}$$

$$\le \|f'\|^2_{L^2(U)} + C\left[\|v_m(0)\|^2_{L^2} + \int_0^T\|f'\|^2_{L^2}\right]$$

LSG

Theorem

Let u_m be the soln of wave equation. Then

$$\int_0^T \|u_m''(t)\|^2_{H^{-1}(\Omega)} \le 2c^2\int_0^t \|u_m\|^2_{H_0^1(\Omega)} + 2c^2\int_0^t \|f(t)\|^2\,dt$$

Proof

let $v \in H_0^1(\Omega)$ and

$v = w + z$

$w \in \operatorname{span}\{w_1, \dots, w_K\} = V_m$

$z \in V_m^{\perp} \Rightarrow (z, w_k) = 0.$

$\|v\| \le \|w\| + \|z\|$. $\partial C L'$

$\|\cdot\|_1 \le \|\cdot\|_L$. $V_m \subset H_0^1$.

$z \in V_m \Rightarrow z \in H_0^1 \Rightarrow \|z\| \le \|V_m\|$

$\cancel{\langle \sum a_i w_i, w_k\rangle}$

$\langle v, v\rangle = \cancel{\langle w, v\rangle} +$

$$\Rightarrow \|w\|_{H_0^1(\Omega)} \le \|v\|_{H_0^1(\Omega)}$$

Choose v as test f^n in weak formulation

$$(u_m'', v) = (u_m'', w+z)$$

$$= (u_m'', w) + \underbrace{(u_m'', z)}_{=0}$$

$$= (u_m'', w) \qquad u_m'' = \sum_{k=1}^{m} d_m^{k''} w_k$$

$$= -c^2(\nabla u_m, \nabla w) + (f, w)$$

$$|(u_m'', v)|$$

$$\le c\left(\|u_m\|_{H_0^1(\Omega)} + \|f\|_{L^2(\Omega)}\right)\|w\|_{H_0^1(\Omega)}$$

$$\le \left(c\|u_m\|_{H_0^1(\Omega)} + \|f\|_{L^2(\Omega)}\right)\|v\|_{H_0^1(\Omega)}$$

$$\Rightarrow \|u_m''\| \le c\left(\|u_m\|_{H_0^1(\Omega)} + \|f\|_{L^2(\Omega)}\right)$$

Integrating and using Cauchy-sch

$$\int_0^T \|u_m''(t)\|^2_{H^{-1}(\Omega)} \le 2c^2\int_0^t \|u_m(t)\|^2_{H_0^1(\Omega)} + 2c^2\int_0^t \|f(t)\|^2\,dt$$

from previous calculations

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$$= 2c^2 \left\{ e^t \left\{ \|\nabla g\|^2_{L^2(\Omega)} + \|h\|^2_{L^2(\Omega)} + \int_0^t \|f(s)\|^2 ds + 2c^2 \int_0^t \|f(s)\|^2 ds \right\}\right.$$

$$= c(t) \left(\|\nabla g\|^2 + \|h\|^2 + \int_0^t \|f(s)\|^2 ds \right)$$

$\exists$ a subsequence, denoted as $u_m \ni m \to \infty$ for

$u_m \to u$ weakly in $L^2(0,T; H_0^1(\Omega))$

$u_m' \to u'$ " $L^2(0,T; L^2(\Omega))$

$u_m'' \to u''$ " $L^2(0,T; H^{-1}(\Omega))$

Theorem

$f \in L^2(Q_T)$, $g \in H_0^1(\Omega)$, $h \in L^2(\Omega)$

then u is soln of wave eqn.

$$\|u\|_{L^\infty(0,T; H_0^1(\Omega))} + \|u'\|_{L^\infty(0,T; L^2(\Omega))} + \|u''\|_{L^2(0,T; H^{-1}(\Omega))} \le C$$

Proof

Suppose $v \in L^2(0,T; H_0^1(\Omega))$

$$v = \sum_{K=1}^{\infty} d_K(t) w_K$$

Multiply the weak formulation with d_K and sum over $K = 1, 2, \dots, m$ we get

$$(u_m'', v) + c^2(\nabla u_m, \nabla v) = (f, v)$$

Integrate over $(0,T)$ and as $m \to \infty$

$$\int_0^T \langle u'', v\rangle_* + c^2(\nabla u, \nabla v) = \int_0^T (f, v)$$

$\Rightarrow u \in C([0,T]; L^2(\Omega))$.

choose, $v \in C^2([0,T]; H_0^1(\Omega))$ $\quad (u'v)\Big|_0^T - \int_0^T u'v' = u'(T)v(T) - u'(0)v(0) - \int_0^T$

$v(T) = v'(T) = 0$.

From weak formulatn,

$$\int_0^T \langle u'', v\rangle_* + c^2(\nabla u, \nabla v) = \int_0^T (f, v)$$

$$\Rightarrow \int_0^T \langle u, v''\rangle + c^2(\nabla u, \nabla v)$$

$$= \int_0^T (f, v)\, dt + (u'(0), v(0)) - (u(0), v'(0)) \quad \text{--- A.}$$

Integrate over $(0,T)$, integration by parts, $m \to \infty$

$$\int_0^T \langle u_m, v''\rangle + c^2(\nabla u_m, \nabla v)$$

$$= \int_0^T (f, v)\, dt + (u_m'(0), v(0)) - (u_m(0), v'(0))$$

$$\to \int_0^T \langle u, v''\rangle + c^2(\nabla u, \nabla v) = \int_0^T (f, v) + (h, v(0)) - (g, v'(0)) \quad \text{--- B}$$

Compare A & B

$(u'(0), v(0)) = (h, v(0))$

$(u(0), v'(0)) = (g, v'(0))$

Uniqueness.

Thm

Soln of wave eqn. is unique.

let $\exists\ u_1, u_2$.

$u_{1tt} - \Delta u_1 = f \quad u_1(0) = g$

$\qquad\qquad u_1'(x,0) = h$

$u_{2tt} - \Delta u_2 = f \quad u_2(x,0) = g$

$\qquad\qquad u_2'(x,0) = h$

$u = u_1 - u_2 \qquad u(x,0) = 0$

$u_{tt} - \Delta u = 0. \qquad u'(x,0) = 0$

$\langle u_{tt}, v\rangle + \langle \nabla u, \nabla v\rangle = 0$

taking $v = u_t$ will give

$\nabla u_t \rightarrow$ existence is not proved.

So the method cannot be used.

let $v(t) = \begin{cases} \int_t^s u(\tau)d\tau & 0 \le t \le s \\ 0 & s \le t \le T \end{cases}$

$\Rightarrow v(t) \in H_0^1(\Omega) \quad \forall\, 0 \le t \le T$

$\int_0^s (u''(t), v(t))_* + c^2 (\nabla u(t), \nabla v(t))\, dt = 0$

$v(s) = 0$

$v'(t) = -u(t)$

$\int_0^s (u'', v(t))_*$

$= (u'(0)\, v(0) - u'(s)\, v(s))$

$- \int_0^s (u'(t), v'(t))\, dt$

$= -\int_0^s (u'(t), -u(t))\, dt$

$= \int_0^s \frac{1}{2} \frac{d}{dt} (\|u(t)\|^2)$

$\int_0^s (\nabla u(t), \nabla v(t))\, dt$

$= \int_0^s (-\nabla v', \nabla v(t))\, dt$

$= \frac{1}{2} \int_0^s \frac{d}{dt} \|\nabla v(t)\|^2\, dt.$

$\int_0^s \frac{1}{2} \frac{d}{dt} (\|u(t)\|^2 - \|\nabla v(t)\|^2)\, dt = 0.$

$\Rightarrow \|u(s)\|^2_{L^2} - \underbrace{\|\nabla v(s)\|^2_{L}}_{=0}$

$- \|u(0)\|^2_{L^2} + \|\nabla v(0)\|^2_{L^2(\Omega)} = 0$

$\overset{\text{"}}{0}$

$\|u(s)\|^2 + \|\nabla v(0)\|^2 = 0$

$\Rightarrow \underline{\underline{u \equiv 0}}.$

NB.

$g \in H_0^1(U) \cap H^2(U)$.

$f \in H^1(0,T;L^2(U))$

Then

$u \in L^\infty(0,T;H^2(U))$

$u' \in L^\infty(0,T;L^2(U)) \cap L^2(0,T;H_0^1(U))$

$u'' \in L^2(0,T;H^{-1}(U))$

$B[u_m, w_k] = (f - u_m', w_k)$

$\beta \|u\|^2_{H^2(U)} \leq \langle Lu, -\Delta u\rangle + \gamma \|u\|^2_{L^2}$

$-\gamma\|u_m\|^2_{L^2} + \beta\|u_m\|^2_{H^2(U)}$

$\leq \langle Lu_m, -\Delta u_m\rangle$

$= (f - u_m', -\Delta u_m)$

$= (f, -\Delta u_m) + (-u_m', -\Delta u_m)$

$\leq \varepsilon\|\Delta u_m\|^2_{L^2} + C(\varepsilon)\|f\|^2_{L^2}$

$\sup\left(\|u_m'\|^2_{L^2} + \|u_m\|^2_{H^2}\right) + \int_0^T \|u_m'\|_{H_0^1(U)}$

$\leq C\left[\|f\|^2_{H^1(0,T;L^2(U))} + \|g\|_{H^2(U)}\right]$

let $v = w + z$

$w \in V_m = \{w_1, w_2, \cdots, w_k\}$

$(u_m'', v) = (u_m'', w + z)$

$= (u_m'', w) + (u_m'', z)$

$= (u_m'', w)$.

$(v_m', w_k) + B[v_m, w_k] = (f', w_k)$

let $w_k = v$

$\langle u_m'', v\rangle = (f', v) - B[u_m', w]$

$|\langle u_m'', v\rangle| \leq \|f'\|\,\|w\|_{H_0^1}$

$+ \|u_m'\|_{H_0^1}\|w\|_{H_0^1}$

$\|u_m''\|_{H^{-1}} \leq \|f'\|_{L^2} + \|u_m'\|_{H_0^1}$;

$\|w\|_{H_0^1} \leq 1$

$\int_0^T \|u_m''\|_{H^{-1}(U)} \leq \|f'\|_{L^2(0,T;L^2(U))} + \|u_m'\|_{L^2(0,T;H_0^1(U))}$

ref. Evans.

- Maximum Principles

$\frac{1}{(4\pi t)^{N/2}} e^{-\frac{|x-\xi|^2}{4t}}$

- Harnack's inequality.

$\frac{u(x_1, t_1)}{u(x_2, t_2)} = \left(\frac{t_2}{t_1}\right)^{N/2} e^{\frac{|x_2-\xi|^2}{4t_2}} e^{-\frac{|x_1-\xi|^2}{4t_1}}$

$\frac{x_2 - \xi}{t_2} = \frac{x_2 - x_1 + x_1 - \xi}{t_2 - t_1 + t_1}$

$u(x_1, t_1) \leq \left(\frac{t_2}{t_1}\right)^{N/2} e^{\frac{|x_2 - x_1|^2}{4(t_2 - t_1)}} u(x_2, t_2)$

Assume $u \in C_1^2(U_T)$, $u_t + Lu = 0$ in U_T, $u \geq 0$ in U_T, $V \subset\subset U$ connected. Then for each ~~$0<t_1$~~ $0 < t_1 < t_2 < T$.

$\exists$ a constant $C \ni$

$$\sup_V u(\cdot, t_1) \leq \inf_V u(\cdot, t_2)$$

where to use Harnack inequality?

$V = \log u$

$V_t = \frac{1}{u} u_t$

$= \frac{1}{u} \left[\sum a^{ij} D_{ij} u \right]$

$D_i V = \frac{1}{u} D_i u$

$D_{ij} V = \frac{1}{u} D_{ij} u - \frac{1}{u^2} D_i u D_j u$

$= \frac{1}{u} D_{ij} u - D_i V D_j V$

$$V_t = \underbrace{\sum a^{ij} D_{ij} V}_{w} + \underbrace{\sum a^{ij} D_i V D_j V}_{\tilde{w}}$$

$\tilde{w} \geq \theta |Du|^2$

$w_t - \sum a^{ij} D_{ij} w + \sum b^i D_i w + R_2 = 0$

~~$w = \sum a^{ij}(x) D_{ij} V$~~

~~$w_t = \sum a^{ij}(x,t) D_{ij} V_t$~~

~~$= \sum a^{ij}(x) D_{ij} \frac{1}{u} u_t$~~

$W = \sum a^{ij}(x,t) D_{ij} V$

$W_t = \sum a^{ij}(x,t) D_{ij} V_t + \sum a^{ij}_t D_{ij} V$

$V_t = w + \tilde{w}$

$\Rightarrow W_t = \sum a^{ij}(x,t) D_{ij} w + \sum a^{ij}(x,t) D_{ij} \tilde{w}$

$+ \sum a_t^{ij} D_{ij} V$

$D_i w = \sum a_i^{kl} D_{kl} V + \sum a^{kl} D_{kli} V$

$\tilde{w} = \sum_{kl} a^{kl} D_k V D_l V$

$D_i \tilde{w} = \sum_{kl} a_i^{kl} D_k V D_l V +$

$\sum_{kl} a^{kl} D_{ki} V D_l V +$

$\sum_{kl} a^{kl} D_k V D_{li} V$

$\sum a^{ij} D_{ij} \tilde{w}$

$= \sum_{ij} a^{ij} \Big[\sum_{kl} a_{ij}^{kl} D_k V D_l V$

$+ \sum_{kl} a_i^{kl} D_{kj} V D_l V$

$+ \sum_{kl} a_i^{kl} D_k V D_{lj} V$

$+ \sum_{kl} a_j^{kl} D_{ki} V D_l V + \sum_{kl} a^{kl} D_{kij} V D_l V$

$+ \sum_{kl} a^{kl} D_{ki} D_{lj} V + \sum_{kl} a_j^{kl} D_k V D_{li} V +$

$\sum_{kl} a^{kl} D_{kj} V D_{li} V + \sum_{kl} a^{kl} D_k V D_{lij} V.$

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$$w_t - \sum a^{ij} D_{ij} w + \sum b^i D_i w$$

$$= 2\sum_{ijkl} a^{ij} a^{kl} D_{ik}V D_{lj} V + R$$

$$b^i = -2\sum a^{ij} D_j V.$$

$$\tilde{w}_t - \sum a^{ij} D_{ij}\tilde{w} + \sum b^i D_i \tilde{w}$$

$$\geq -c|D^2 V|^2 - c|DV|^2 - c$$

let $\hat{w} = w + \alpha\tilde{w}$.

LSG.

$$\left.\begin{aligned} u_{tt} - Lu &= f \text{ in } Q_T \\ u(x,t) &= 0 \text{ on } \Sigma_T. \end{aligned}\right\} hp$$

$f \in L^2(Q_T)$

$g \in H_0^1(\Omega)$

$h \in L^2(\Omega).$

$a_{ij}(x,t) \in C^1(Q_T)$

$b_i(x,t) \in C^1(Q_T)$ } C^1 is needed for uniqueness.

$c(x,t) \in C^1(Q_T)$

$$Lu = \sum_{i,j=1}^{n} (a_{ij} u_{x_i})_{x_j} + \sum_{i=1}^{n} b_i u_{x_i} + cu$$

Bilinear form

$$B[u,v] = \int_\Omega \sum_{ij} a_{ij} u_{x_i} v_{x_j} + \sum b_i u_{x_i} v + cuv\, dx$$

$\forall v \in H_0^1(\Omega)$. $0 \le t \le T$.

Weak soln

We say a fn $u \in L^2(0,T;H_0^1(\Omega))$, $u' \in L^2(0,T;L^2(\Omega))$, $u'' \in L^2(0,T;H^{-1}(\Omega))$ is a weak soln of (hp) if it satisfies,

i) $\langle u'', v\rangle_* + B(u,v) = (f,v)$ $\forall v \in H_0^1(\Omega)$

ii) $u(x,0) = g$, $u_t(0) = h$. $0 \le t \le T$.

(fixed point method o Faedo-Galerkin can be used to prove existence)

Faedo-Galerkin

$$u_m = \sum_{k=1}^{m} d_k(t)\, w_k.$$

$d_k(0) = (g, w_k)$. $u_m(0) = g$

$d_k'(0) = (h, w_k)$ $u_m'(0) = h$

We look to find $d_k(t)$, $k = 1,2,\dots m$

$$\langle u_m'', w_k\rangle + B[u_m, w_k] = (f, w_k)$$

$$\langle u_m'', w_k\rangle = \langle \sum_{k=1}^{m} d_k'' w_k, w_k\rangle$$

$$= d_k''(t).$$

$B[u_m, w_k]$

$$= B\left[\sum_{k=1}^{m} d_k(t) w_k, w_k\right]$$

$$= \int_\Omega a_{kk}\, d_k(t) |Dw|^2$$

$$+ \sum_{i=1}^{m} b_i\, d_k(t) \nabla w_k\, w_k$$

$$+ c \sum^{m} d_k(t) w_k\, w_k.$$

$B[u_m, w_K]$

$$= \sum_{l=1}^{m} B[w_l, w_K; t]\, d_l(t)$$

$$= \sum_{l=1}^{m} e^{Kl}\, d_l(t)$$

where $e^{Kl} = B[w_l, w_K; t]$

$$f^K = (f, w_K)$$

$$d_K''(t) + \sum_{l=1}^{m} e^{Kl}(t)\, d_l(t) = f^K$$

$$d_K(0) = (g, w_K)$$

$$d_K'(0) = (h, w_K).$$

This is a 2nd order IVP

$\Rightarrow \exists$ a unique soln

$$u_m = \sum_{K=1}^{m} d_K(t)\, w_K$$ is approximate soln of (hp).

$$u_m(0) = g$$

$$u_m'(0) = h.$$

<u>Energy estimates</u>

<u>Thm.</u> $\exists$ a constant $c > 0$ depends on Ω, T and a_{ij}, b_i, c of L $\ni$

$$\max_{0 \le t \le T} \left(\|u(t)\|_{H_0^1(\Omega)} + \|u'\|_{L^2} + \|u_m''\|_{L^2(0,T; H^{-1}(\Omega))} \right)$$

$$\le C(\Omega, T, a_{ij}, b_i, c) \left[\|\nabla g\|_{H_0^1(\Omega)} + \|h\|_{L^2(\Omega)} + \|f\|_{L^2(0,T; L^2(\Omega))} \right]$$

<u>Proof</u>.

$$(u_m'', u_m') + B[u_m, u_m'] = (f, u_m')$$

$$\langle u_m'', u_m' \rangle = \frac{1}{2} \frac{d}{dt} \|u_m'\|^2_{L^2(\Omega)}$$

$$B[u_m, u_m'] = \int_\Omega \underbrace{\sum a_{ij}\, u_{m x_i} u'_{m x_j}}_{B_1} \underbrace{+ \sum b_i\, u_{m x_j} u_m' + c\, u_m u_m'}_{B_2}\, dx.$$

$\frac{1}{2}\frac{d}{dt}$

Let us consider symmetric bilinear form,

$$A[u, v] = \int_\Omega \sum_{ij} a_{ij}\, u_{x_i} v_{x_j}$$

$$\# \ \frac{1}{2} \frac{d}{dt} A[u_m, u_m]$$

$$= \frac{1}{2} \frac{d}{dt} \int_\Omega a_{ij}\, u_{m x_i} u_{m x_j}$$

$$\frac{1}{2} \left[\int_\Omega \sum a_{ij}'\, u_{m x_i} u_{m x_j} \right] + \underbrace{\int_\Omega \sum a_{ij}\, u_{m x_i} u'_{m x_j}}_{B_1} \quad (\because a_{ij} = a_{ji})$$

$$B_1 = \frac{d}{dt}\left[\frac{1}{2} A[u_m, u_m]\right]$$

$$- \frac{1}{2}\int_\Omega \sum a_{ij}' \, u_{mx_i} u_{mx_j}$$

$$B_1 \geq \frac{d}{dt}\left[\frac{1}{2} A[u_m, u_m]\right]$$

$$- \underbrace{c(a_{ij}')}_{\text{const.}} \|u_m\|^2_{H_0^1(\Omega)}$$

$$B_2 \leq \|b_i\| \, \|\nabla u_m\|_{L^2(\Omega)} \|u_m'\|_{L^2(\Omega)}$$

$$+ \|c\| \, \|u_m\|_{L^2(\Omega)} \|u_m'\|_{L^2(\Omega)}$$

$$\leq c(\|b_i\|, \|c\|)$$

$$* \left(\frac{1}{2}\|u_m\|^2_{H_0^1(\Omega)} + \|u_m'\|^2_{L^2(\Omega)}\right).$$

$$\frac{1}{2}\frac{d}{dt}\left[\|u_m'\|^2_{L^2(\Omega)} + A[u_m, u_m]\right]$$

$$\leq c\left(\|u_m'\|^2 + \|u_m\|^2_{H_0^1(\Omega)} + \|f\|^2_{L^2(\Omega)}\right)$$

$$-④$$

Uniform hyperbolicity condit^n,

$$\theta \int_\Omega \|\nabla u\|^2 \leq \int_\Omega \sum a_{ij} u_{x_i} u_{x_j}$$

$$= A(u, u) \quad -⑤$$

④ & ⑤ $\Rightarrow$

$$\frac{1}{2}\frac{d}{dt}\left[\|u_m'\|^2_{L^2(\Omega)} + c\|u_m\|^2_{H_0^1(\Omega)}\right]$$

$$\leq c\left(\|u_m'\|^2_{L^2} + \|u_m\|^2_{H_0^1(\Omega)} + \|f\|^2_{L^2(\Omega)}\right)$$

$$\eta(t) = \|u_m'\|^2_{L^2(\Omega)} + \|u_m\|^2_{H_0^1(\Omega)}$$

$$\psi(t) = \|f\|^2_{L^2(\Omega)}$$

$$\eta'(t) \leq c_1 \eta(t) + c_2 \psi(t)$$

$$\Rightarrow \eta(t) \leq e^{c_1 t}\left(\eta(0) + c_2\int_0^t \psi(s)\,ds\right)$$

$$\|u_m'\|^2_{L^2(\Omega)} + \|u_m\|^2_{H_0^1(\Omega)}$$

$$\leq e^{c_1 t}\left(\|u_m'(0)\|^2_{L^2(\Omega)} + \|u_m(0)\|^2_{H_0^1(\Omega)}\right.$$

$$\left. + c_2\int_0^t \|f\|^2_{L^2(\Omega)}\,ds\right)$$

$$\doteq c(c_1 T)\left(\|h\|^2_{L^2(\Omega)} + \|g\|^2_{H_0^1(\Omega)}\right.$$

$$\left. + \int_0^t \|f\|^2_{L^2(\Omega)}\,ds\right)$$

$\Rightarrow$

$$\max_{0 \leq t \leq T} \|u_m'\|^2_{L^2(\Omega)} + \max_{0 \leq t \leq T} \|u_m\|^2_{H_0^1(\Omega)}$$

$$\leq c(c_1 T)\left(\|h\|^2_{L^2(\Omega)} + \|g\|^2_{H_0^1(\Omega)}\right.$$

$$\left. + \int_0^t \|f\|^2_{L^2(\Omega)}\,ds\right)$$

$$u_{tt} - \nabla a(t, \nabla u) = f. \quad f \in L^2, f \in L^1$$

$\rightarrow$ a satisfies monotonicity condition

Renormalized

$u_m \to u$ in $L^2(0,T; H_0^1(\Omega))$

$u_m' \to u'$ in $L^2(0,T; L^2(\Omega))$

$u_m'' \to u''$ in $L^2(0,T; H^1(\Omega))$

let $v(t) = \sum_{k=1}^{N} d_k(t)\, \omega_k$

Multiply weak formulation by d_k & sum over $k=1,2,\dots n$ and integrate over $(0,T)$

$$\int_0^T \langle u_m'', v\rangle + B[u_m, v] = \int_0^T (f, v). \quad \text{---(1)}$$

$m \to \infty$

$$\int_0^T \langle u'', v\rangle + B[u,v] = \int_0^T (f,v) \quad \text{---(2)}$$

$$\Rightarrow \langle u'', v\rangle + B[u,v] = (f,v)$$

$\forall\, v \in H_0^1(\Omega)$

$t \in (0,T)$

<u>claim</u>

$u(0) = g$; $u'(0) = h$.

$V(T), V'(T) = 0$.

$$-\int_0^T \langle u_m', v'\rangle\, dt + v \cdot u_m' \Big|_0^T + \int_0^T B[u_m, v] = \int_0^T (f,v)$$

$$\Rightarrow -\int_0^T \langle u_m', v'\rangle dt + v(T) u_m'(T) - v(0) u_m'(0) + \int_0^T B[u_m, v] = \int_0^T (f,v)$$

$$\Rightarrow \int_0^T \langle u_m, v''\rangle dt + v'(0) u_m(0) - v(0) u_m'(0) + \int_0^T B[u_m, v] = \int_0^T (f,v)$$

$m \to \infty$

$$\int_0^T \langle u_m, v''\rangle dt + g\, v'(0) - v(0) h + \int_0^T B[u,v] = \int_0^T (f,v) \quad \text{---(3)}$$

(2) $\Rightarrow$

$$\int_0^T (u, v'') + B[u,v] = \int_0^T (f, v)\, dt + (u'(0), v(0)) - (u(0), v(0)).$$

let $v(t) = \begin{cases} \int_t^s u(\tau)\, d\tau & 0 \le t \le s \\ 0 & s \le t \le T \end{cases}$

$v(t) \in H_0^1(\Omega)$, $0 \le t \le T$.

$u = u_1 - u_2$, $g = h = f = 0$.

$$\int_0^s (u'', v(t)) + B(u, v(t)) = 0 \quad \cancel{\int_\Omega (f,v)}$$

$v(s) = 0$

$v'(t) = -u(t)$

Integration by parts,

$$-\int_0^s (u', v') + B[u,v] = 0$$

$$\int_0^s (u', u) - B(v', v) = 0. \quad \text{---(1)}$$

$$\frac{1}{2}\frac{d}{dt} B[v,v]$$

$$= \frac{1}{2}\int a_{ij}' v_{x_i} v_{x_j} + \frac{1}{2}\int_\Omega b_i' v_x v + \frac{1}{2}\int_\Omega c' v \cdot v + \int_\Omega a_{ij} v_{x_i}' v_{x_j}$$

$$+ \frac{1}{2} b_i v_{x_i}' v + c v' v + \frac{1}{2} b_i v_x v'$$

$$+ \left(\frac{1}{2} b_i v_x' v - \frac{1}{2} b_i v_x' v\right)$$

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Circled terms $= D[v, v]$

$= D[v, v] + B[v', v] - C[v, v]$ —(2)

(2) in (1).

$$\int_0^s \frac{d}{dt}\|u\|^2_{L^2(\Omega)} - \frac{1}{2}\frac{d}{dt}B[v,v]$$

$$= \int_0^s D[v,v] - C(v,v)$$

$v'(0) = v(s) = 0$.

$$\frac{1}{2}\left(\|u(s)\|^2_{L^2(\Omega)} + B(v(0), v(0)\right)$$

$$\leq C\int_0^s \|v\|^2_{H_0^1(\Omega)} + \|v\|_{L^2(\Omega)}\, dt$$

$$+ \cancel{\|v(0)\|^2_{L^2(\Omega)}}.$$

$$\alpha \|u\|^2_{H_0^1(\Omega)} \leq B[u,u] + \beta\|u\|^2_{L^2(\Omega)}$$

$$\Rightarrow \frac{1}{2}\left(\|u(s)\|^2_{L^2(\Omega)} + \alpha\|v(0)\|^2_{H_0^1(\Omega)}\right)$$

$$\leq C\int_0^s \|v\|^2_{H_0^1(\Omega)} + \|v\|^2_{L^2(\Omega)}\, dt$$

$$+ \|v(0)\|^2_{L^2(\Omega)}$$

let $w(t) = \int_0^t u(\tau)\,d\tau = v(0)$

and $\alpha = 1$ $\Rightarrow$ ~~$v(0)$~~

$$\|u(s)\|^2_{L^2(\Omega)} + \|w(s)\|^2_{H_0^1(\Omega)}$$

$$\leq C\int_0^s \|v\|^2_{H_0^1(\Omega)} + \|v\|^2_{L^2(\Omega)}\, ds$$

$$+ \|w(s)\|^2_{L^2(\Omega)}$$

$$-(w(t) - w(s)) = v(t)$$

$$\|w(s) - w(t)\|^2 \leq 2\|w(t)\|^2_{H_0^1(\Omega)} + 2\|w(s)\|^2_{H_0^1(\Omega)}$$

$$\|w(s)\|^2_{L^2(\Omega)} \leq \int_0^s \|u(\tau)\|^2_{L^2(\Omega)}\, ds$$

$$\|u(s)\|^2_{L^2(\Omega)} + \|w(s)\|^2_{H_0^1(\Omega)}$$

$$\leq 2C\int_0^s \|w(t)\|^2_{H_0^1(\Omega)}\, dt + 2C\int_0^s \|w(t)\|\, dt$$

$$+ \|u(t)\|^2\, dt$$

$$\|u(s)\|^2_{L^2(\Omega)} + (1 - 2Cs)\|w(s)\|^2$$

$$\leq C\int_0^s \|w(t)\|^2 + \|u(s)\|^2\, ds.$$

T is sufficiently small so that

$1 - 2CT_1 \geq 0$.

Gronwall's inequality

$$\|u(s)\|^2_{L^2(\Omega)} + \|w(s)\|^2_{H_0^1(\Omega)} = 0$$

$\Rightarrow u = 0$ in $[0, T_1]$.

Similarly extend for

$[T_1, 2T_1]$, $[2T_1, 3T_1]$ ...